CONSTRUCTION PRINCIPLES AND CLASSIFICATION OF MATHEMATICAL MODELS

Zhanar Omarova
Associate Professor of IITU
Department of Information Systems
and Mathematical Modeling

Main goal of all external world discovery of investigations should be the rational order and harmony which God gave to the world and revealed it to us in the language of mathematics.

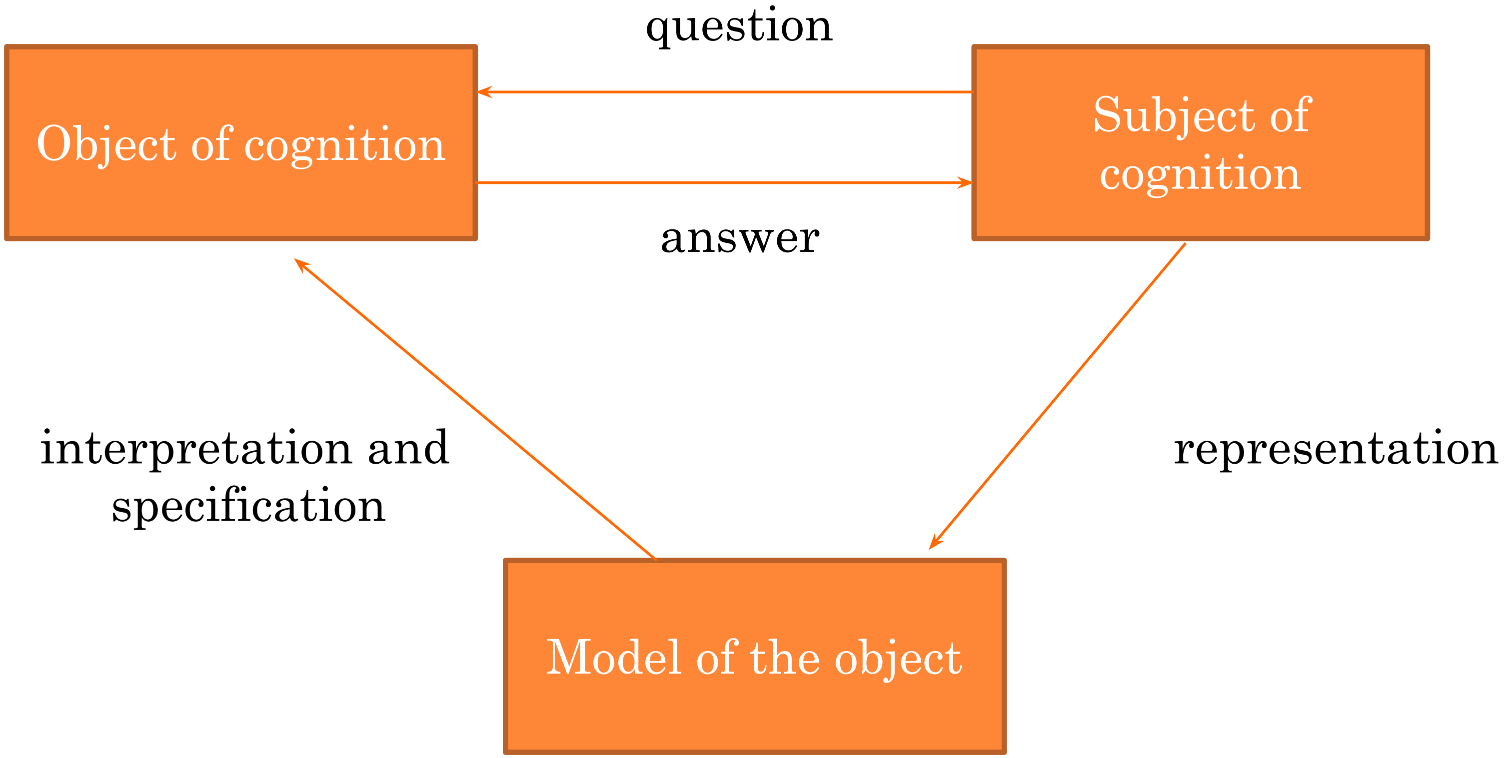
Johannes Kepler

- Mathematics is radically different from physics, biology, history, psychology and other sciences. Its object are the abstract concepts like numbers and functions, equations and sets, which are strange products of the human brain, however mathematics is surprisingly adapted to comprehend all the phenomena of nature and society.
- Striking applicability of mathematics for analyzing various world events is the result of the link between objective reality and the abstraction of the mathematical structures.

- ***Mathematical modeling is a specific form of the world cognition, which is able to translate the laws of nature studied by individual sciences into mathematical language, to see real-life events behind the strict mathematical formulas.***
- Our goal is to learn how to connect mathematics and the world, make sure that ***different natural phenomena can be described mathematically with the same methods.*** There is strict mechanics, thoroughly imbued with mathematical ideas, and psychology, which is not very amenable to formalization in amazingly equal position.

- Researcher studies objects of interest, watches the course of events, and interferes in them by asking certain questions and getting appropriate answers. After examining some information about the object of the study, the subject forms his idea about the objects of interest and his view of the phenomenon. The view which is based on the available objective information about the object under study but representing a subjective point of view of the researcher is called *a model* of the object.
- Thus, the process of cognition, which is essentially reduced to the collection, storage and processing of all kinds of *information*, is at the same time the process of modeling.

Simplified diagram of the cognition process



MAIN STAGES OF MATHEMATICAL MODELING

№	stage	goal	means
1	model building	mathematical description of the process	laws of nature established by the private science
2	model analysis	detection of the information contained in the model in a latent form	mathematical and computer analysis of model
3	model identification	check on the adequacy of the model	comparison of the results of the model analysis and experimental data
4	application of the model	use of the model properties to get information about the process	solving of prediction, control, etc. problems

PRINCIPLES OF THE MATHEMATICAL MODEL CONSTRUCTION

- It is possible to identify some common laws in the construction of mathematical models. First of all, we should understand what exactly is the object of study.
- After that it is necessary to identify *functions of the system condition (or system functions)* which are the values which according to the researcher fully reflect the progress of the process. Mathematical models usually represent equations for the selected system functions.
- Exploring the resulting mathematical model and solving the relevant equations we try to understand type of dependence of the system functions on various parameters. First of all, these are *independent variables*, which are usually time and space variables.

- After determination of the system functions and independent variables, we have to specify *the coordinate system* within which a mathematical model will be constructed.
- Now we can try to go directly to the construction of mathematical relations that characterize the law of variation of the system functions. We must find out what exactly influenced the course of events and specify *the reason of the system evolution*.
- After indication the reasons that led to the events which are of our interest, we must identify *the cause-and-effect relationship* and determine how exactly each of these reasons influenced the simulated process. Basis of mathematical model is cause-and-effect relationship which expresses the laws of variation of the system functions. It is formed on the basis of underlying regularities established by particular science.

- Having written mathematical relations which express cause-and-effect relation for the phenomenon, we will find out that besides the system functions and independent variables mathematical model includes characteristics of completely different nature. We are talking about *the parameters of the system*, allowing to choose the particular case which is of direct interest to us at this stage of the study.
- Formally, the system parameters can be changed arbitrarily. However, not all the variants of setting these values have physical meaning. Conditions of applicability of the mathematical model, range of the parameters (independent variables and parameters of the system), in which a model makes sense should be specified.

- Thus, the mathematical model usually includes three classes of characteristics. Parameters of the process in each case are fixed quantities which compose *the input data*. Independent variables are not fixed, they change within a certain range and form the domain of the system functions. Thus, the mathematical model is most often a problem of restoring the system functions dependence on the corresponding variables in the allowed set of process parameters.
- After determination necessary mathematical relations, we have to find what information about the process is practically interesting for us besides the system functions. We have to specify *the output information*.

Process of construction mathematical model includes the following elements:

- Object of study
- Functions of the system condition / System functions
- Independent variables
- Coordinate system
- Reasons for the system evolution
- Cause-and-effect relations
- Input parameters of the system
- Conditions of the mathematical model applicability
- Output parameters of the system

- Now we have specific mathematical relations. They should be studied on the basis of various quantitative and qualitative methods. Conducting appropriate analysis using only mathematical tools (usually computer), we return to the studied phenomena in order to make physical interpretation of the results, refine our tasks and summarize relevant outcomes.

CLASSIFICATION OF MATHEMATICAL MODELS

Depending on the specifics of variables constituting mathematical models, we can specify some of the principles of classification.

- *Lumped parameter system,* in which the system function depends only on time.
- *Distributed parameter systems,* which allow dependence of the system function on space variables.

Ordinary differential equations most often represent lumped parameter systems. Distributed parameter systems are usually characterized by *partial differential equations*.

- *Stochastic models,* which may have impact on the process random factors.
- *Deterministic models*, in which such impact is ignored, and system state is unambiguously determined.
- *Continuous systems* in which independent variables are changing continuously.
- *Discrete systems* with independent variables that vary in pace (fixed or not).
- *Dynamic systems* with the system functions that change with the time.
- *Stationary systems* which characteristics do not change over time.

Equation of the falling body

- The first mathematical model, which meets all academic requirements is associated with the name of Galileo and describes the process of falling bodies under their own weight.
- When the body is falling, its distance to the ground will be change over time. Thus, we can describe the process using function $y = y(t)$ which characterizes the height of the body at any given time t.
- Thus, the motion of the body is necessarily accompanied by changing its height. In order to find the required function y, we can estimate velocity of its change.

- If the velocity v of the body is known, we obtain the relation

$$= v \qquad (\dot{y}1)$$

Equation (1.1), we suppose, should describe the process under study, as its mathematical model.

The second stage of the study involves the extracting using mathematical tools information which is contained in the resulting model in a latent form. To find the unknown functional relation we have to solve the resulting equation. If the velocity does not change over time, equation (1.1) will be satisfied by any function of the following form:

$$y(t) = vt + c, \qquad (1.2)$$

where c is an arbitrary constant. Relation (1.2) gives the general solution of the equation (1.1).

- In the third stage of the study we will try to interpret the results. First of all, we have to find out whether constructed model describes the phenomenon in question. To do this, we compare the results of analysis of the mathematical model and experiment.
- The ambiguity of the found solution means that the information about the process is still not enough for a full description. Therefore, proposed mathematical model needs considerable refinement. This means that we have to return to the first step and try to fill a gap in the understanding of the process under study.

- Obviously, the position of the body will essentially depend on the exact position of the body at some initial time. Suppose that body has initial height:

$$y(t_0)=y_0 \qquad (1.3)$$

It is called *the initial condition*.

- Refined mathematical model is characterized by equations (1.1), (1.3), which constitute the *Cauchy problem* for the equation. We define a solution of the equation (1.1), which satisfies the initial condition (1.3). We put in the formula (1.2) $t = t_0$.

$$y(t_0)=vt_0+c=y_0$$

- From this we find the constant: $c = y_0 - vt_0$. Then the solution of equation (1.1) with the initial condition (1.3) has final form:

$$y(t) = y_0 + v(t-t_0) \qquad (1.4)$$

- We again interpret the results and find out mismatch of calculated height of falling body to its experimental value. Indeed, a simple experiment shows that in the period of falling the body velocity is changing. Specifying the model, we get the same Cauchy problem, but with a variable v.
- Integrating equation (1.1) from t_0 to some value t and using the initial condition (1.3), we find the following law of body height change:

$$y(t)=y_0+ \qquad (1 \int_{t_0}^{t} v(\tau)d\tau$$

- In the case of constant velocity, this formula takes the familiar form (1.4).

- The variation law of the body velocity is not known. Velocity of change of the function v, that is its derivative $\dot{v}$ (or second derivative of the coordinate $\ddot{y}$) in mechanics is called *acceleration* and is denoted by ***a***. As a result, to find the velocity we obtain another differential equation.

$$= a \qquad \dot{v} \qquad (1.6)$$

- We can assume, for example, that the body at the time $t = t_0$ just starts to move, and therefore it has zero velocity:

$$v(t_0)=0 \qquad (1.7)$$

- So, we have a system of differential equations (1.1), (1.6) with initial conditions (1.3) and (1.7).

- Definition of the body height is reduced to finding its derivative (velocity). For calculation of velocity is required to find its derivative (acceleration). Reliable experiment shows that the acceleration of the falling body does not change over time, and it does depend on the specific characteristics of the body under study, as a universal physical constant. This striking fact allows us, finally, to complete the research and get the result.
- Note that the falling body height is reduced in the motion, in accordance with equation (1.1) it is only possible in case of the negative velocity v.

- Positive value $g = -a$ is called *free fall acceleration*, and it is an absolute constant which is independent from process conditions. Thus, the relation (1.6) can be written as:

$$\dot{v} = -g \qquad (1.8)$$

- Differential equations (1.1), (1.8) with initial conditions (1.3), (1.7) are the mathematical model of the free falling process of the body.

- Solving equation (1.8) with the initial condition (1.7), we find that the velocity of a falling body will be changed by the following law:

$$v(t)=-gt$$

Substituting this value in equation (1.5) with $t_0=0$, we establish the following dependence of the falling body height on time for various values of initial height

$$\boldsymbol{y(t)=y_0-gt^2/2}$$

- Comparing the variation law of the falling body height with the experimental results, we conclude that constructed model is quite satisfactory.

The additional information

- Then, based on the properties of the mathematical model we can get some additional information on the behavior of the system under study. In particular, it is important for the process to set the time *T*, in which the body hits the ground, and therefore will be *y(T)* equal to zero.
- We find the value $y_0 = gT^2/2$. From this we can determine the value of the falling body, depending on its initial height

$$T = \sqrt{2y_0/g} \tag{1.9}$$

- Another characteristic of the process, which has practical importance, is the body velocity at the falling time. For a given initial height of the body, it is equal to:

$$v(T) = -\sqrt{2y_0 g}$$

- A final remark concerns the *conditions of applicability* of the mathematical model. The model describes the studied process only in the time interval from zero to T, as long as the body does not reach the ground. Thus, equations (1.1), (1.8) make sense only for a limited time interval $0 < t < T$, where time T is characterized by equation:

$$y(\mathrm{T}) = 0 \qquad (1.10)$$

- Finally, the parameter y_0, included in the initial condition (1.3), must be necessarily positive:

$$y_0 > 0 \qquad (1.11)$$

- Thus, the complete mathematical model of the process includes the state equations (1.1), (1.8) on the time interval $0 < t < T$, the initial conditions (1.3), (1.7), relation (1.9) to find the time *T*, and the condition (1.11) for the system parameter y_0.

MAIN ELEMENTS OF THE MATHEMATICAL MODEL OF THE FALLING BODY PROCESS

№	elements	falling body process
1	object of study	falling body
2	system functions	Height and velocity of the body
3	independent variable	time
4	coordinate system	coordinate is directed vertically up, the point of origin is the surface of the earth
5	reason for the evolution	the force of gravity
6	cause-and-effect relation	the second derivative of the coordinate is proportional to the force of gravity
7	input parameters	initial height of the body
8	conditions of the model applicability	movement until the landing, initial height of the body is positive
9	output parameters	the time of falling, velocity at the time of landing

- It should be noted that the results correspond to some particular level of study of the process, and if necessary can be refined. Thus, we have completely ignored the effect of air resistance, the effect of other forces. We have absolutely no interest in the shape, size and weight, which is also under certain conditions themselves, may change. Actually, the immutability of the free fall acceleration will be recognized as true only for relatively small discontinuity of height.

Thank you for your attention!