NUFYP Mathematics

Lecture 2.1
Exponentials and Logarithms

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Lecture Outline

- Exponent
- Exponential function
- Graphs of exponential functions
- Logarithm
- Graphs of logarithmic functions
- Laws of logarithms

Introduction

What is **exponent**?

What is the **basic idea** of **exponentiation**?

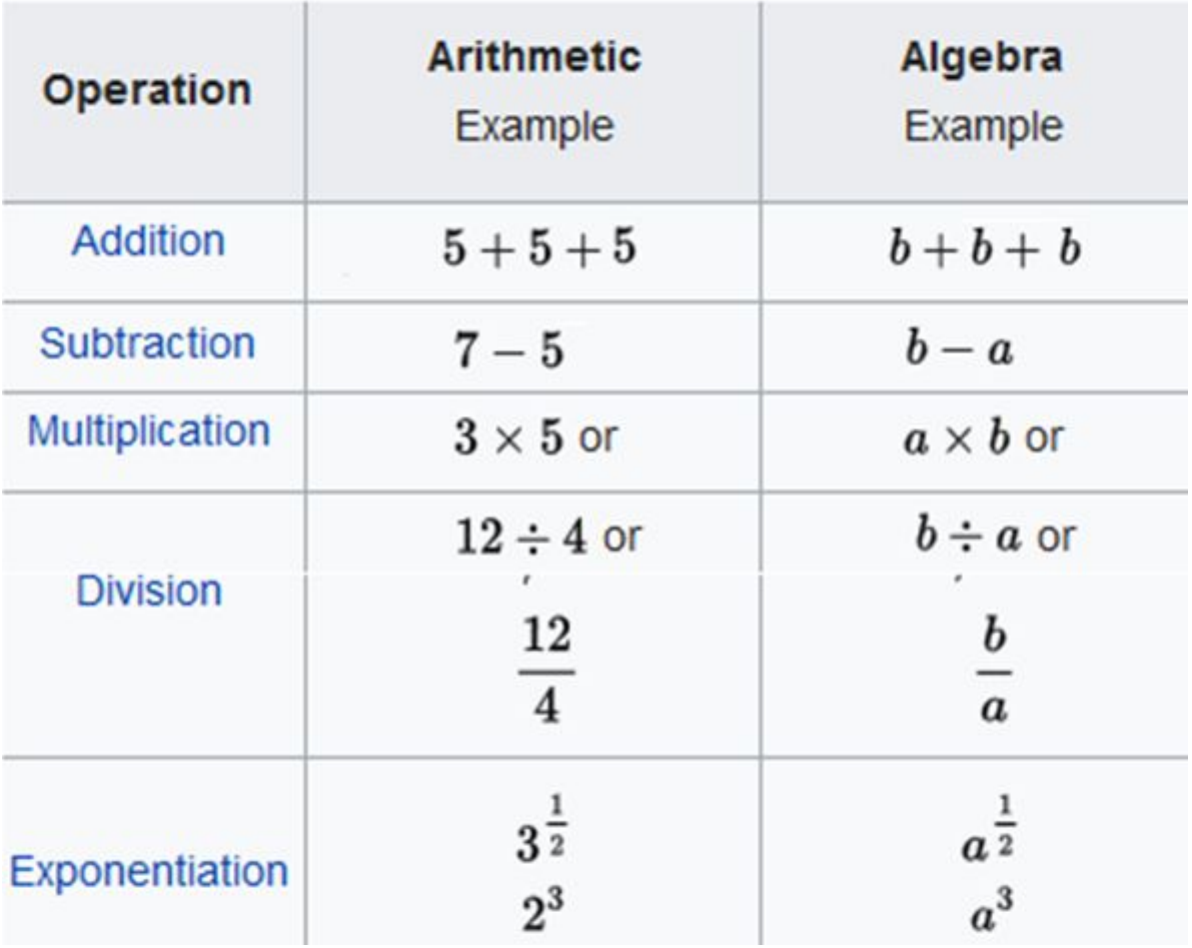
Introduction

What is **exponent**? Exponent is an **index** or **power**.

What is the **basic idea** of **exponentiation**?

Introduction

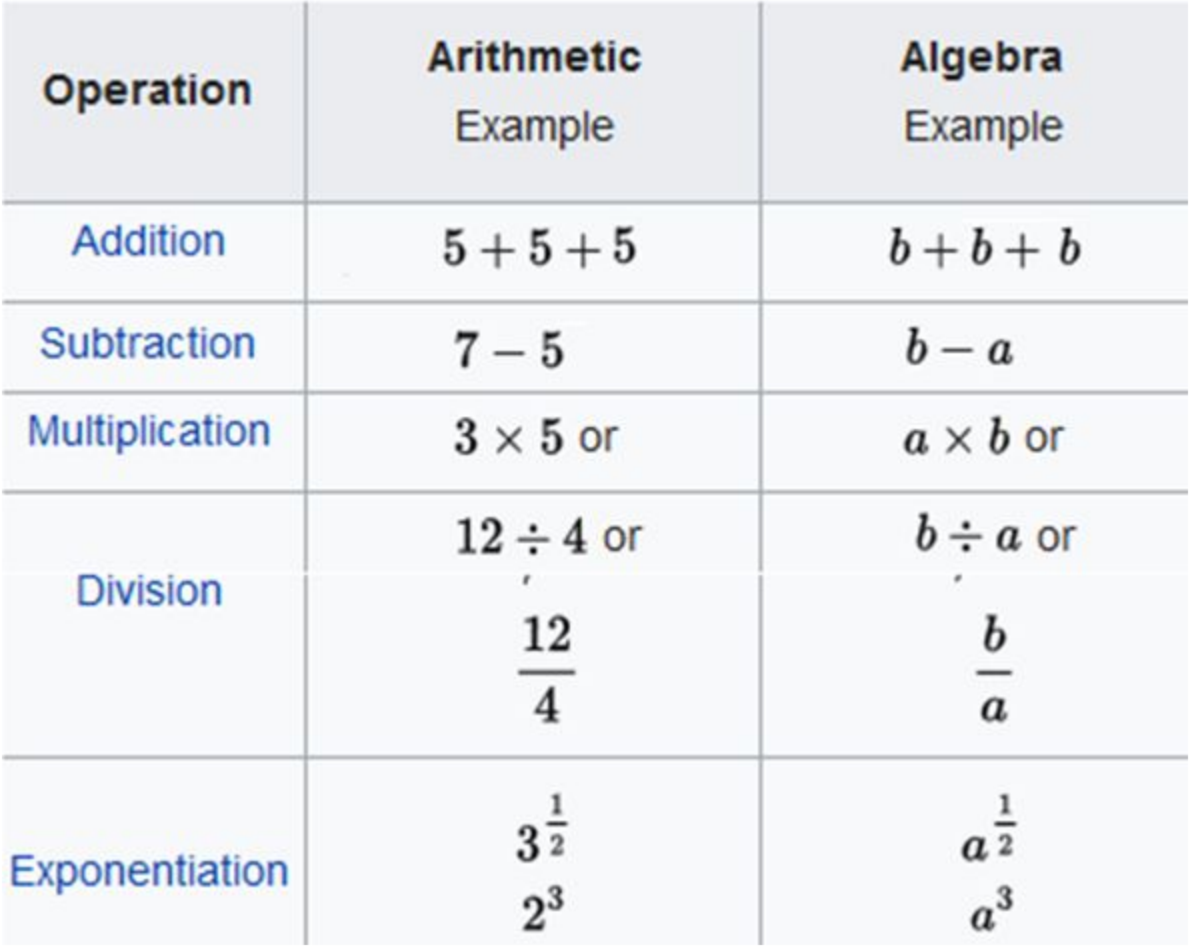
What is the **basic idea** of **exponentiation**?



Operation	**Arithmetic** Example	**Algebra** Example
Addition	$5+5+5$	$b+b+b$
Subtraction	$7-5$	$b-a$
Multiplication	3×5 or	$a \times b$ or
Division	$12 \div 4$ or , $\frac{12}{4}$	$b \div a$ or , $\frac{b}{a}$
Exponentiation	$3^{\frac{1}{2}}$ 2^3	$a^{\frac{1}{2}}$ a^3

Introduction

What is the **basic idea** of **exponentiation**?



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Repeated addition

Repeated multiplication

Exponential function

An ***exponential function*** has the form

Exponent, index, power (**variable**)

$$f(x) = a^x$$

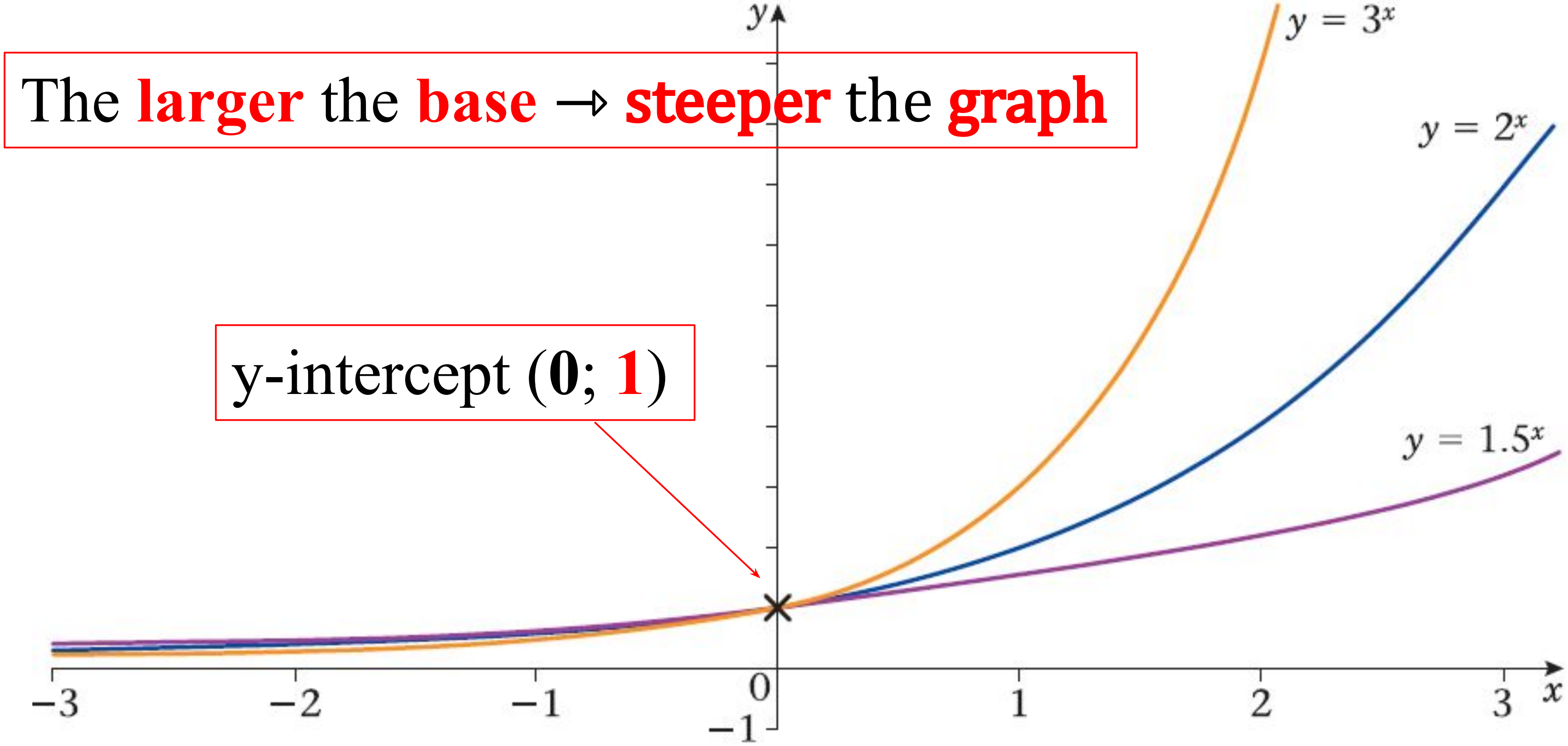
base

- where *a* is constant
- $a \neq 1,\ a > 0$

Examples: $f(x) = 2^x$, $f(x) = 3^x$, $f(x) = e^x$

2.1.1 Sketch the graph of Exponential function

Let us see some graphs of exponential functions with different bases on the same axes:



The **larger** the **base** → **steeper** the **graph**

y-intercept (**0**; **1**)

An ***exponential function*** has the form

Why $a \neq 1$?

Exponent, index, power (variable)

$$f(x) = a^x$$

base

- where a is constant
- $a \neq 1$, $a > 0$

Examples: $f(x) = 2^x$, $f(x) = 3^x$, $f(x) = e^x$

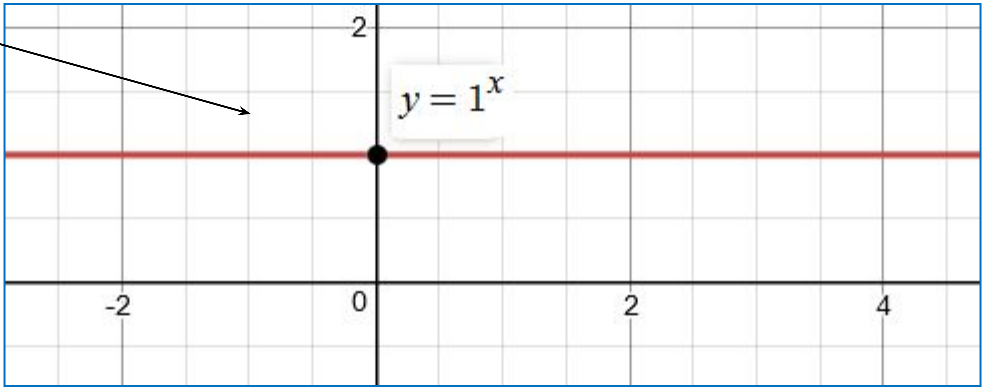
An ***exponential function*** has the form

Exponent, index, power (variable)

When $a = 1$

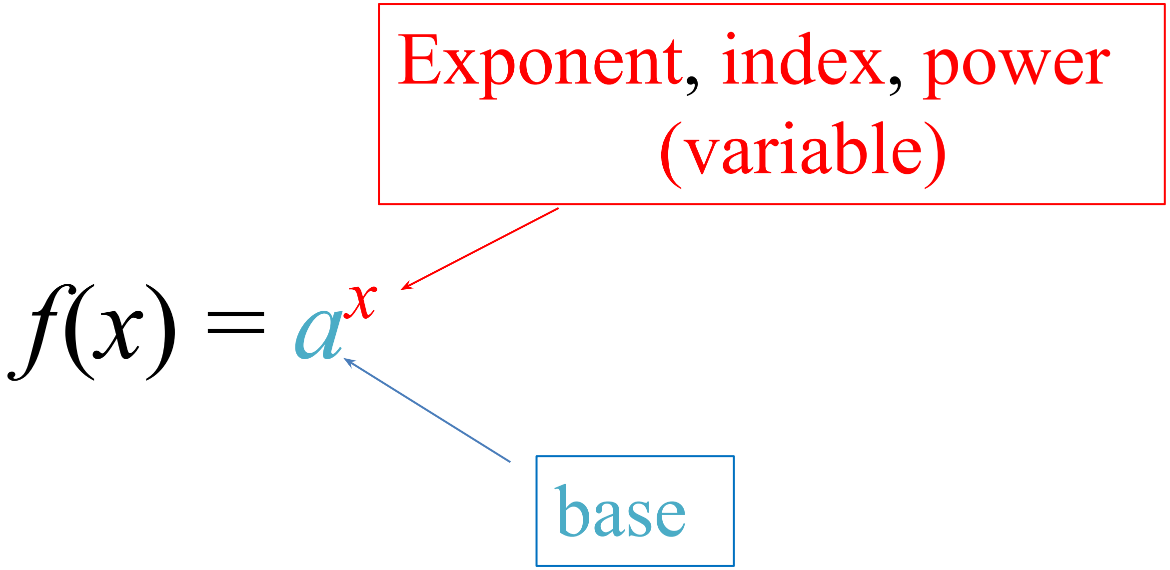
$$f(x) = a^x$$

- where a is constant
- $a \neq 1$, $a > 0$



An ***exponential function*** has the form

Why a >0?



$$f(x) = a^x$$

Exponent, index, power (variable)

base

- where a is constant
- $a \neq 1$, $a > 0$

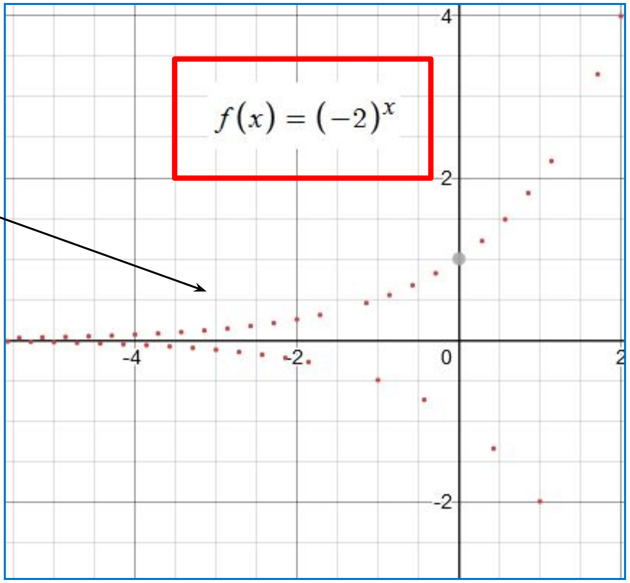
How graph will look like if *a* is negative?

An ***exponential function*** has the form

Exponent, index, power (variable)

Why a >0?

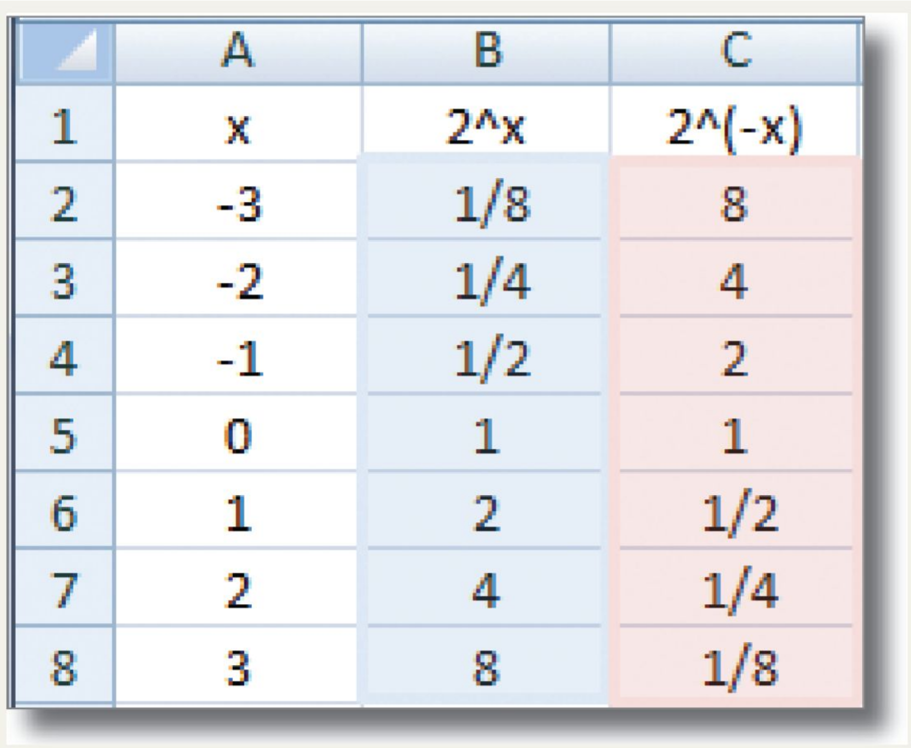
$$f(x) = a^{x}$$



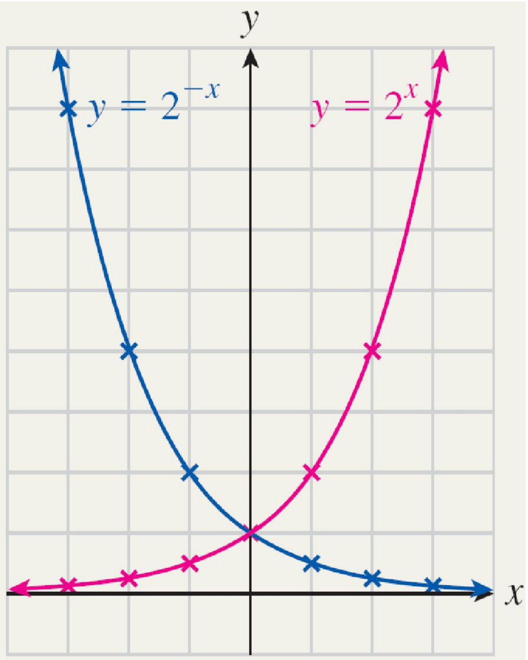
- where a is constant
- $a \neq 1$, $a > 0$

Example:

$$f(x) = 2^x \qquad g(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$$

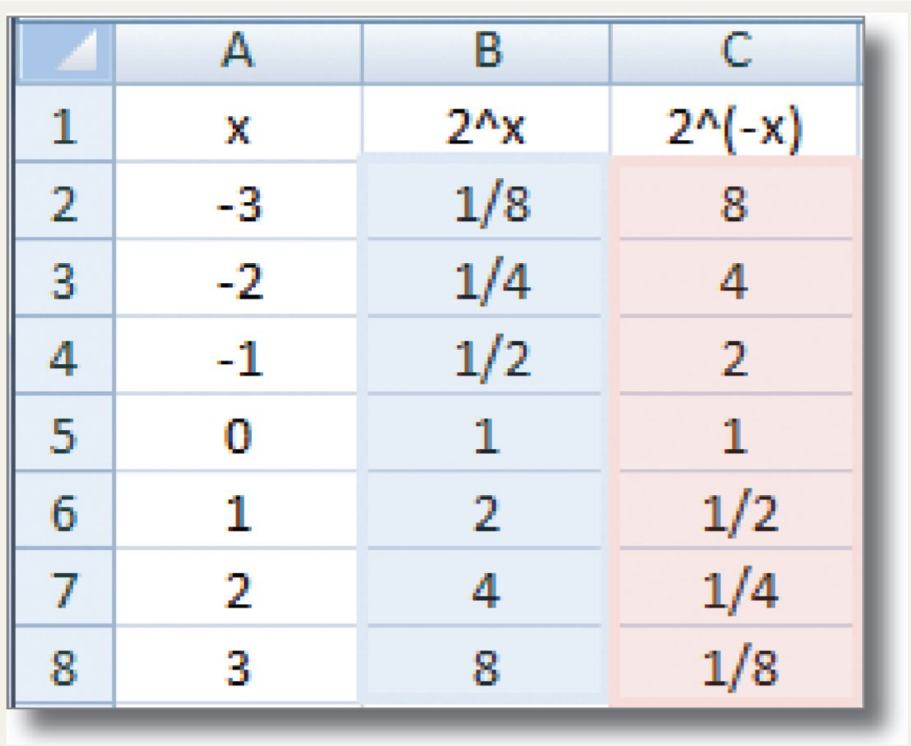


	A	B	C
1	x	2^x	2^(-x)
2	-3	1/8	8
3	-2	1/4	4
4	-1	1/2	2
5	0	1	1
6	1	2	1/2
7	2	4	1/4
8	3	8	1/8



The graph of *g(x)* is a reflection of the graph of *f(x)* over the *y*-axis

Example: $f(x) = 2^x$ $\quad f(-x) = \left(\frac{1}{2}\right)^x = 2^{-x}$



	A	B	C
1	x	2^x	2^(-x)
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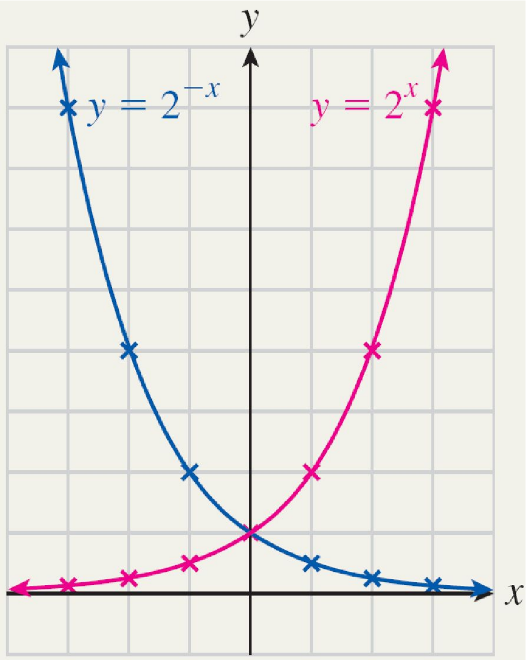
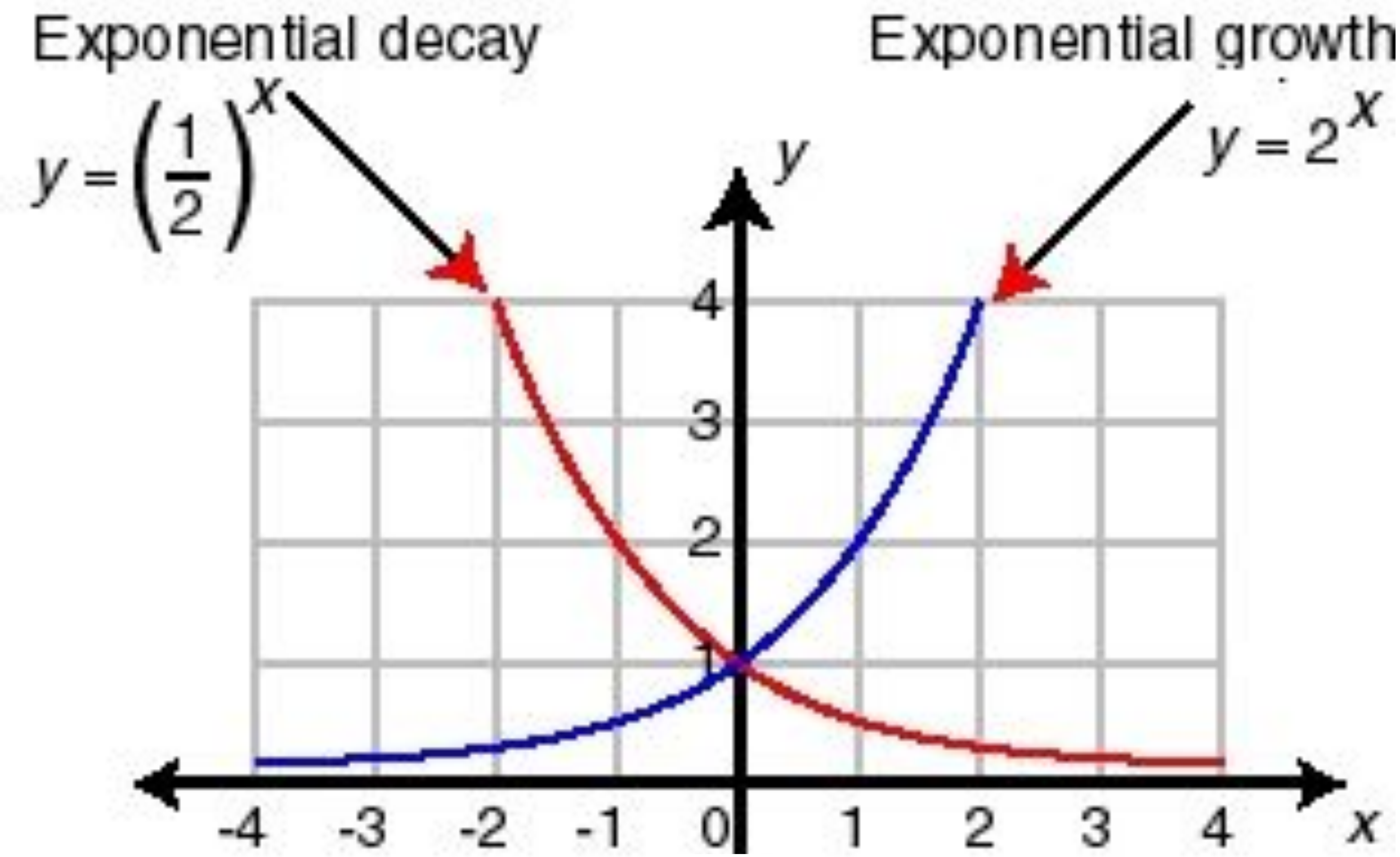
5	0	1	1
6	1	2	1/2
7	2	4	1/4
8	3	8	1/8



The graph of *g(x)* is a reflection of the graph of *f(x)* over the *y*-axis



Exponential Decay

$y = a^x$

$0 < a < 1$

Exponential Growth

$y = a^x$

$a > 1$

An ***exponential function*** has more general form

$$g(x) = Af(x) = Aa^x$$

(0,*A*) *is the y-intercept*

Examples:

$f\ (x)\ =\ 5 * 2^x,\qquad f(x) = 7 * 3^x,\qquad f(x) = 2 * e^x$

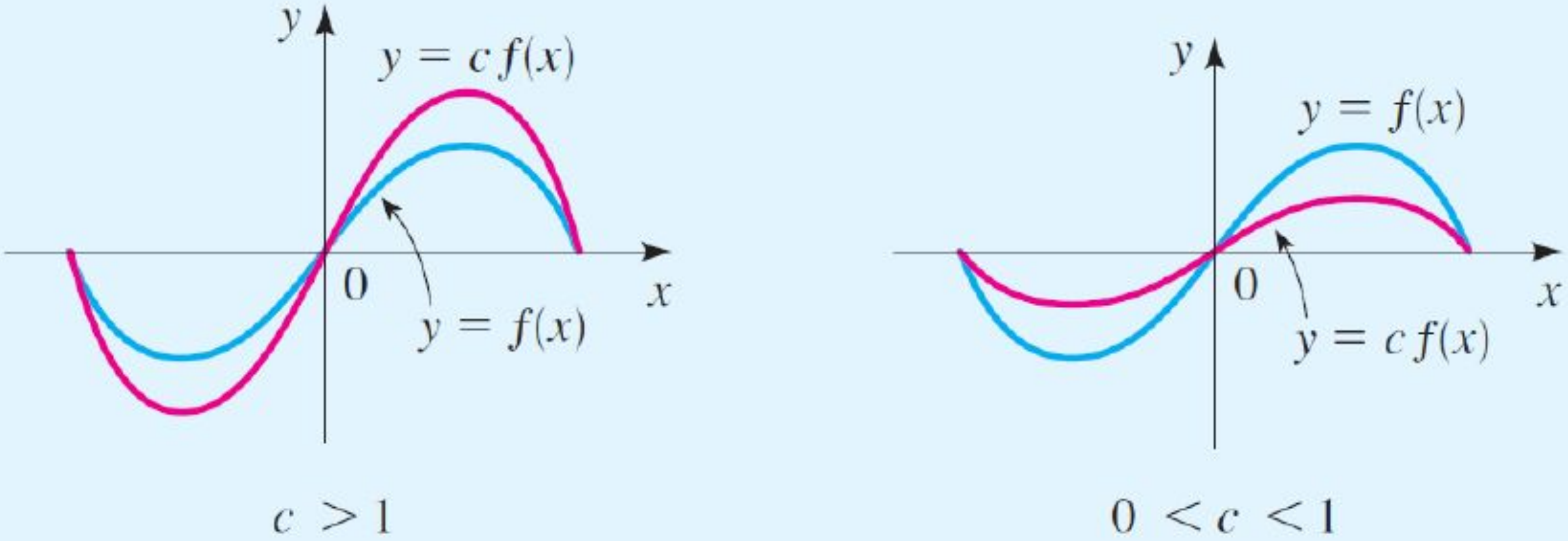
Recall from Lecture 1.5 Vertical scaling

Vertical Stretching and Shrinking of Graphs

To graph $y = cf(x)$:

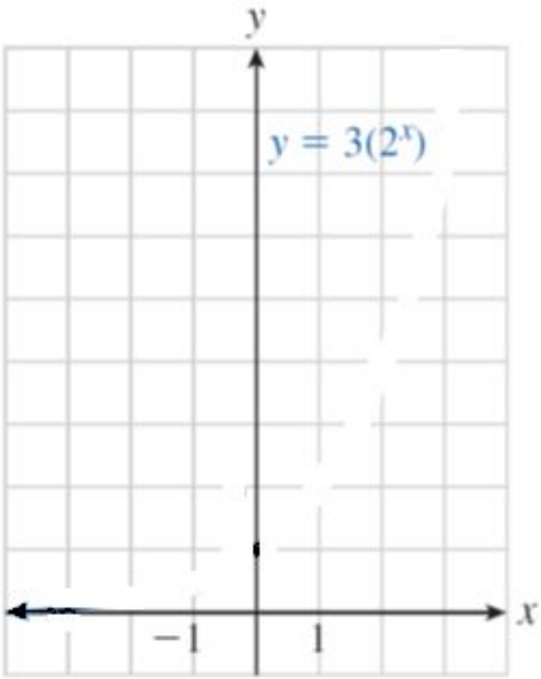
If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.



Example: $f(x) = 2^x$ $h(x) = 3 * 2^x$

$$f(x) = Aa^x$$

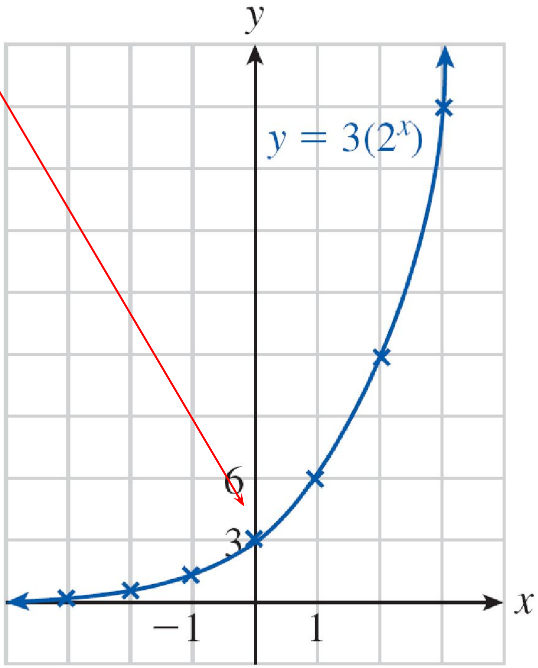


Example: $f(x) = 2^x$ $h(x) = 3 * 2^x$

$h(x) = Aa^x$

x	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{2}$	3	6	12	24

Notice that (0, A)=(0, 3) is the y-intercept



$$g(x) = Aa^{x}$$

$(0,A)$ *is the y-intercept*

What about a?

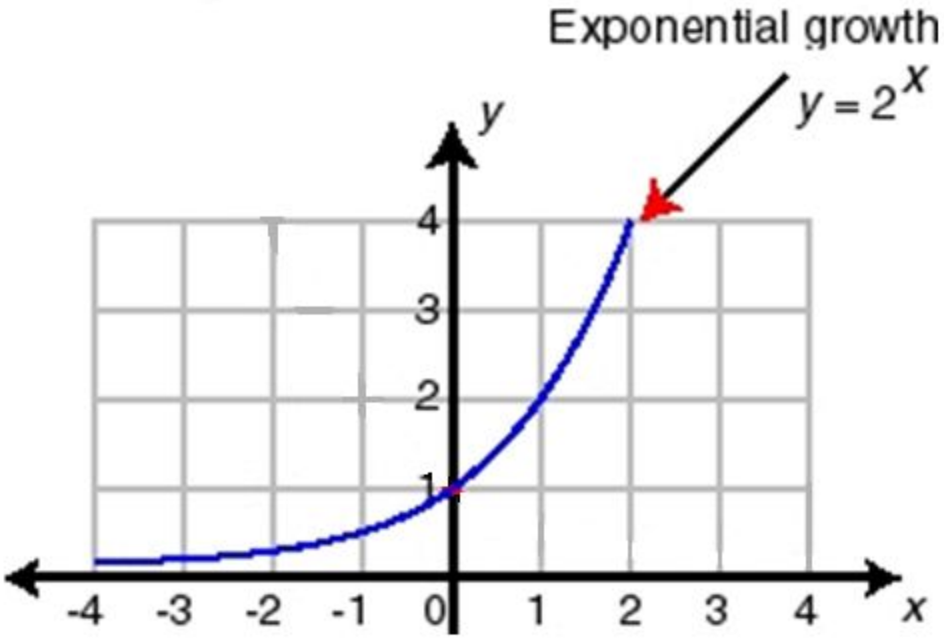
The value of y is multiplied by a for every one-unit increase of x.

Example: $$\boldsymbol{f_1(x) = 2^x}$$

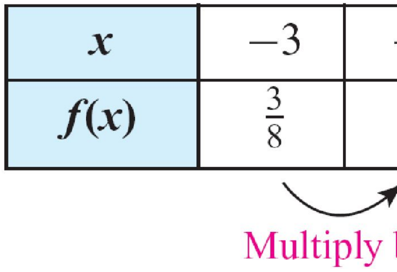
Multiply by 2

Exponential Growth

On the graph, if we move one unit to the right from any point on the curve, the *y* coordinate doubles. Thus, the curve becomes dramatically steeper as the value of *x* increases. This phenomenon is called ***exponential growth***.



Exponential Decay



x	-3
$f(x)$	$\frac{3}{8}$

Multiply

In general: *in the graph of* $f(x) = Aa^x$, $(0, A)$ *is the y-intercept*.

What about a? Notice from the table that the value of y is multiplied by $a = 2$ for every increase of x by 1. If we decrease x by 1, the y coordinate gets ***divided*** by $a = 2$.

Example:

$$f_1(x) = \left(\frac{1}{2}\right)^x$$

Divide by 2

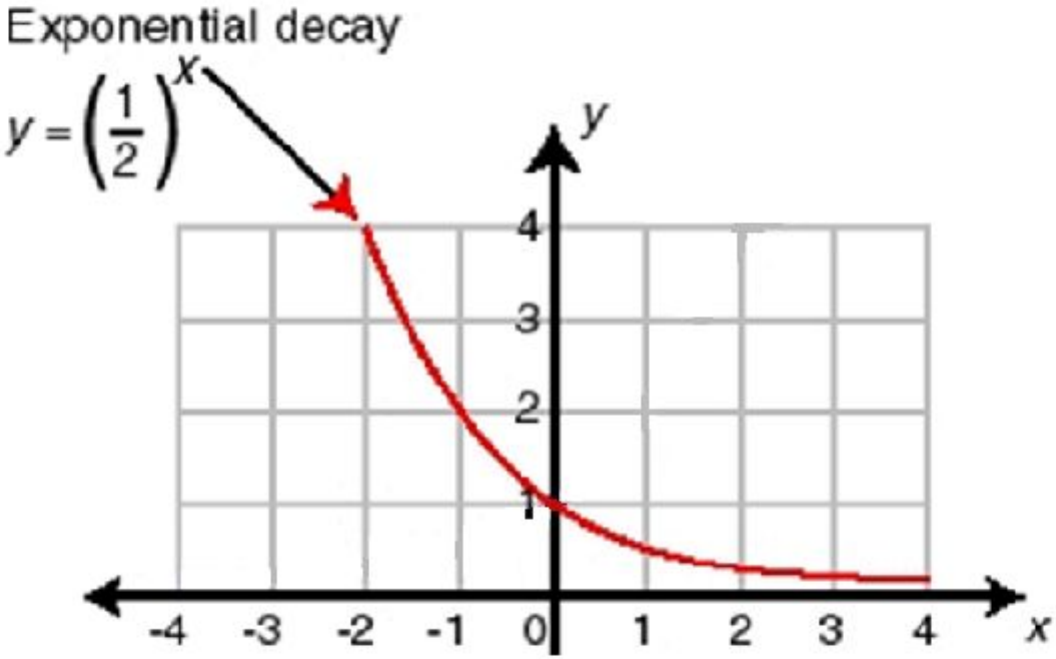
Exponential Decay

When x increases by 1, $f_2(x)$ is multiplied by $\frac{1}{2}$.

The function $f_1(x) = 2^x$ illustrates exponential growth, while

$$f_2(x) = \left(\frac{1}{2}\right)^x$$

illustrates the opposite phenomenon: **exponential decay.**



For **exponential graphs**, the **independent variable** often represents **time** and so in this situation, instead of the letter x, the letter t is usually used.

A quantity y experiences **exponential growth** if $y = Aa^t$ with $a > 1$.

It experiences **exponential decay** if $y = Aa^t$ with $0 < a < 1$.

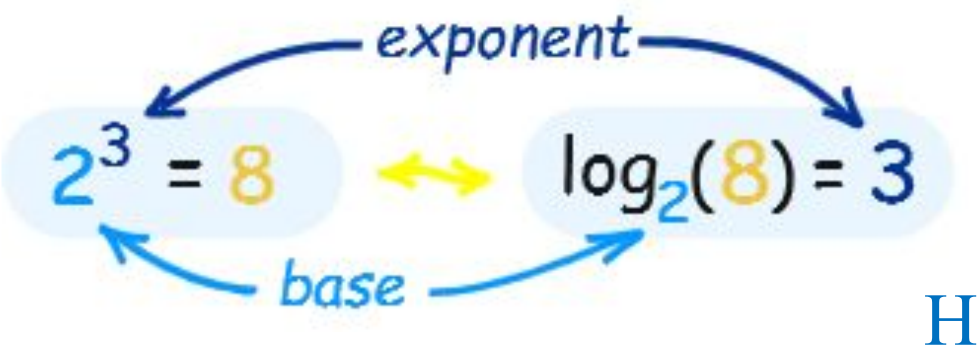
We shall return to this topic in the next lecture and show applications of it to real life context.

2.1.2 Write an expression in logarithmic form

Exponential form vs Logarithmic form

Exponential form

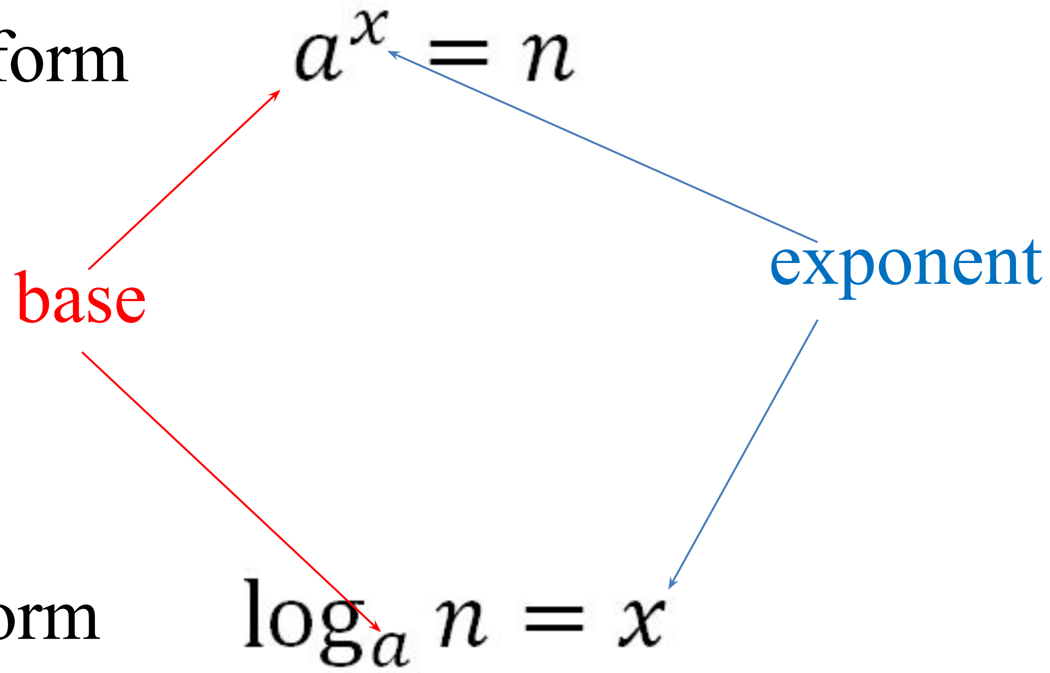
Logarithmic form



If we will multiply base 2 three times by itself what will be the output?

How many times do we need to multiply base 2 by itself to get output been 8?

Exponential form vs Logarithmic form



Exponential form $a^x = n$

base

exponent

Logarithmic form $\log_a n = x$

$a \neq 1 \quad a > 0$

Examples

$\log_{10} 1000 =$

$\log_4 16 =$

$\log_3 27 =$

$\log_5 5 =$

$\log_3 1 =$

$\log_{0.25} 16 =$

$\log_4 (1/16) =$

$\log_{25} 5 =$

Examples

$\log_{10} 1000 = 3$

$\log_{4} 16 = 2$

$\log_{3} 27 = 3$

$\log_{5} 5 = 1$

$\log_{3} 1 = 0$

$\log_{0.25} 16 = -2$

$\log_{4} (1/16) = -2$

$\log_{25} 5 = 1/2$

Common Logarithm

The logarithm with base 10 is called the common logarithm and can be written using one of the following notations:

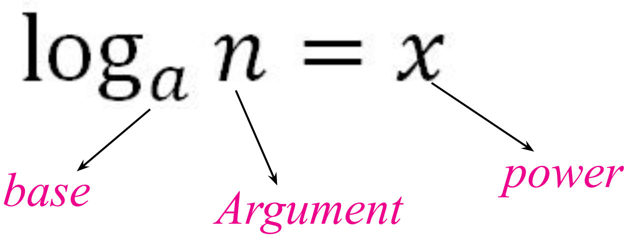
$$\log_{10} x = \log x = \lg x$$

Example

$\log 10000 = 4$

$\log 101 \sim 2{,}0043$

What are the numbers that base of logarithm can be?



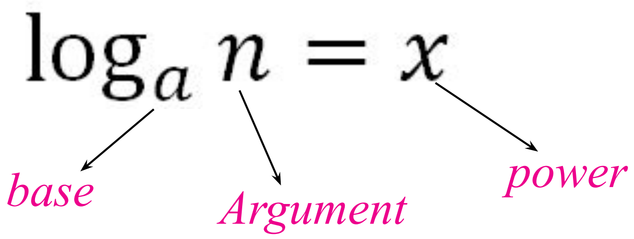
$$\log_a n = x$$

Base of logarithm

Can it be

1) Equal to zero?
2) Equal to 1?
3) Be a Negative number?
4) Be a Positive number?

What are the numbers that base of logarithm can be?



$$\log_a n = x$$

base

Argument

power

Base of logarithm

Can it be

1) Equal to zero?

$\log_0 2 = q$ $\quad 0^q = 2$

$\log_0 5 = w$ $\quad 0^w = 5$

$\log_0 10 = r$ $\quad 0^r = 10$

What are the numbers that base of logarithm can be?

$$\log_{\underset{\textit{base}}{a}} \underset{\textit{Argument}}{n} = \underset{\textit{power}}{x}$$

Base of logarithm

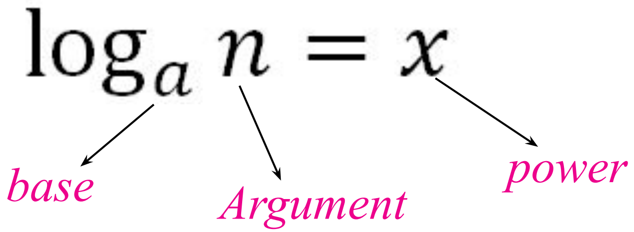
Can it be

1) Equal to zero?

$$\xcancel{\begin{aligned} \log_0 2 &= q & \qquad 0^q &= 2 \\ \log_0 5 &= w & \qquad 0^w &= 5 \\ \log_0 10 &= r & \qquad 0^r &= 10 \end{aligned}}$$

Won’t work, because Zero raised to any power is still zero

What are the numbers that base of logarithm can be?



$$\log_a n = x$$

base

Argument

power

Base of logarithm

Can it be

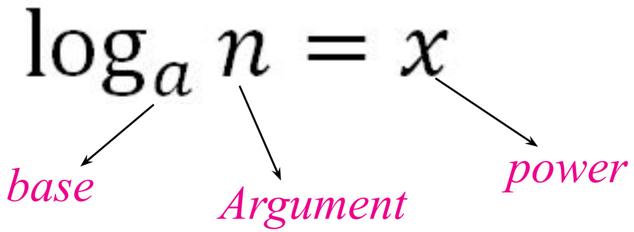
2) Equal to 1?

$$\log_1 2 = q \qquad 1^q = 2$$

$$\log_1 5 = w \qquad 1^w = 5$$

$$\log_1 10 = r \qquad 1^r = 10$$

What are the numbers that base of logarithm can be?



$$\log_a n = x$$

Base of logarithm

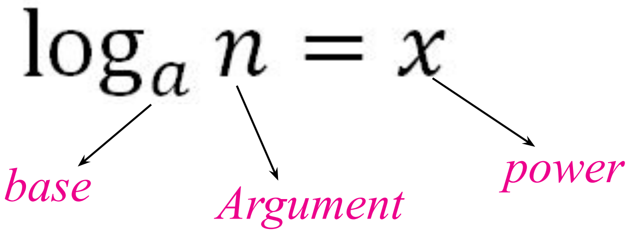
Can it be

2) Equal to 1?

$$\xcancel{\begin{array}{ll} \log_1 2 = q & 1^q = 2 \\ \log_1 5 = w & 1^w = 5 \\ \log_1 10 = r & 1^r = 10 \end{array}}$$

Won’t work, because One raised to the any power is still One

What are the numbers that base of logarithm can be?



$$\log_a n = x$$

base

Argument

power

Base of logarithm

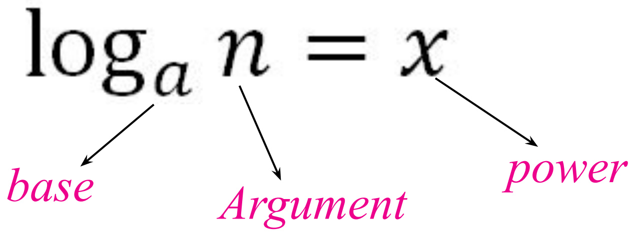
Can it be
3) Be a Negative number?

$$\log_{-2} x = \frac{1}{2}$$

$$(-2)^{\frac{1}{2}} = x$$

$$\sqrt{-2} = x$$

What are the numbers that base of logarithm can be?



$$\log_a n = x$$

Base of logarithm

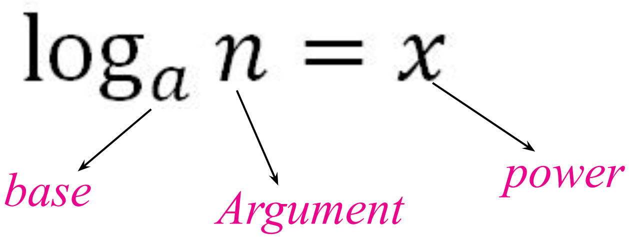
Can it be

3) Be a Negative number?

$$\xcancel{\begin{gathered}\log_{-2} x = \frac{1}{2}\\ (-2)^{\frac{1}{2}} = x\\ \sqrt{-2} = x\end{gathered}}$$

No solution, as we can't take square root of negative number

What are the numbers that base of logarithm can be?



$$\log_a n = x$$

base

Argument

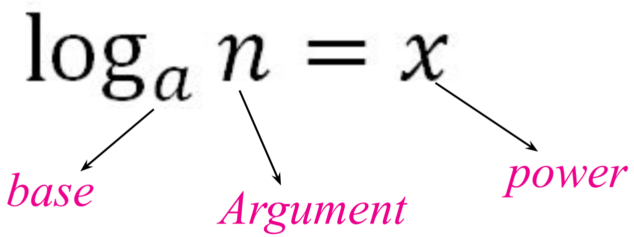
power

Base of logarithm

Can it be

4) Be a Positive number?

What are the numbers that base of logarithm can be?


$$\log_a n = x$$

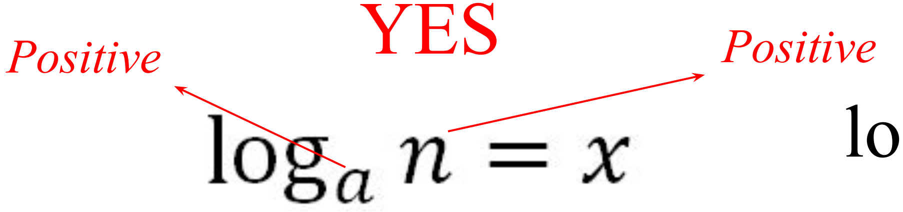
Base of logarithm

Can it be

4) Be a Positive number?

$$positive^x = positive$$

YES


$$\log_a n = x$$

log 10000 =4

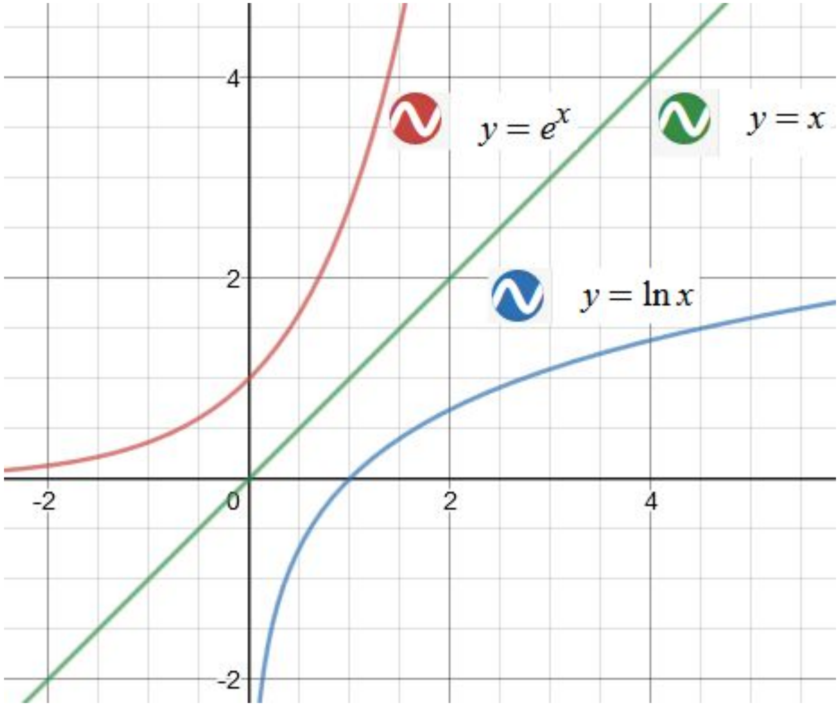
Base been positive will always provide us with positive Argument, no matter what is a value of exponent.

2.1.3 Recognise that the Logarithmic function is an inverse of Exponential function

Since the functions

$f(x)=e^x$ and $g(x)=\ln x$

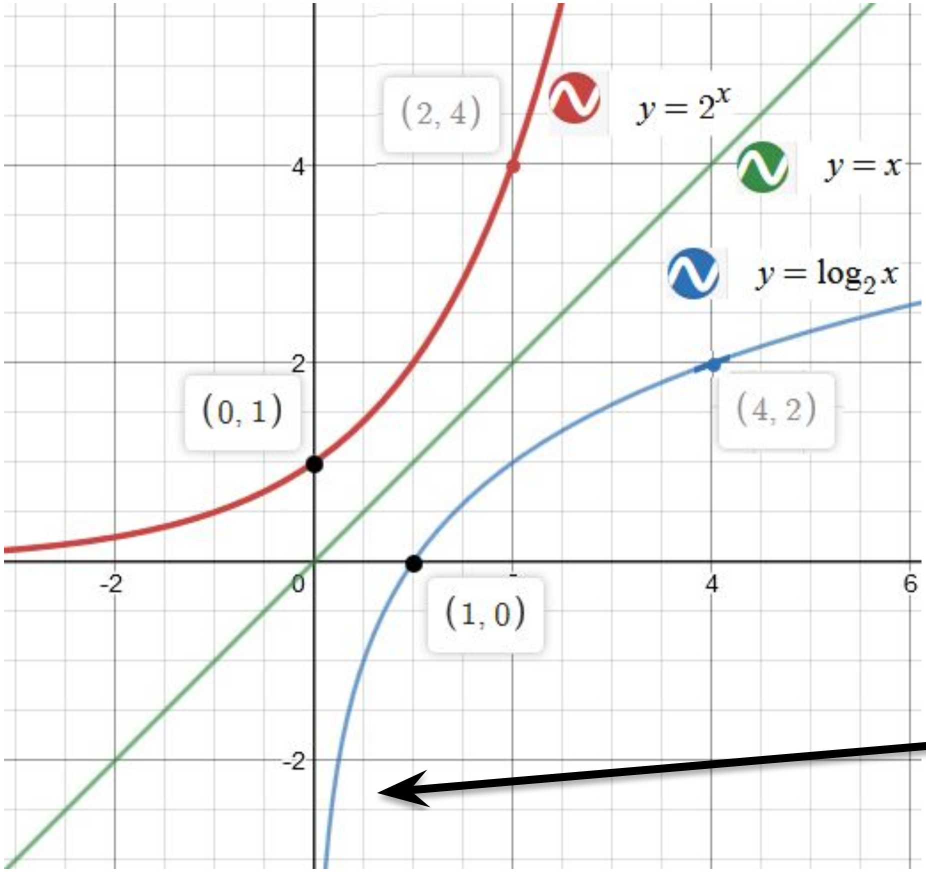
are **inverses** of **each other**, the corresponding graphs are symmetric with respect to the line $y=x$.



2.1.4 Sketch the graph Logarithmic function

Example:

Sketch graphs of $f(x)=2^x$ and $g(x)=\log_2 x$



Horizontal asymptote y=0

Vertical asymptote x=0

Logarithmic Function

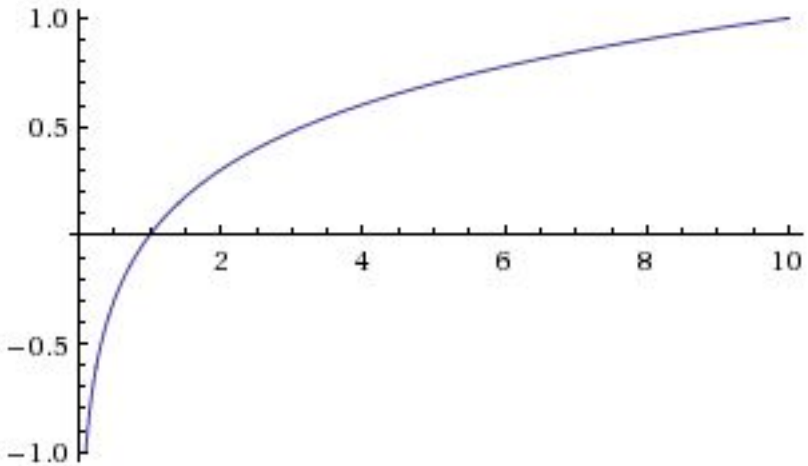
A **logarithmic function** has the form

$$f(x) = a + b * \log_k(cx + d)$$

(b, B and C are constants with $k > 0$, $k \neq 1$)

Quick Examples

$f(x) = \log x$



Computed by Wolfram|Alpha

2.1.5 Apply the laws of logs

Logarithm Identities

The following identities hold for all positive bases $a \neq 1$ and $b \neq 1$, all positive numbers x and y, and every real number r. These identities follow from the laws of exponents.

1. $\log_b(xy) = \log_b x + \log_b y$	$\log_2 16 = \log_2 8 + \log_2 2$
2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	$\log_2\left(\frac{5}{3}\right) = \log_2 5 - \log_2 3$
3. $\log_b(x^r) = r \log_b x$	$\log_2(6^5) = 5 \log_2 6$
4. $\log_b b = 1;\ \log_b 1 = 0$	$\log_2 2 = 1;\ \ln e = 1;\ \log_{11} 1 = 0$
5. $\log_b\left(\frac{1}{x}\right) = -\log_b x$	$\log_2\left(\frac{1}{3}\right) = -\log_2 3$
6. $\log_b x = \frac{\log_a x}{\log_a b}$	$\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} = \frac{\log 5}{\log 2}$

As a sample, let us verify that the first identity holds.

Let

$$\log_a x=b \quad \text{and} \quad \log_a y=c$$

from which we obtain

$$a^b=x \qquad \text{and} \quad a^c=y$$

and therefore

$$xy=a^b.\, a^c=a^{b+c}$$

that allows us to conclude that

$$\log_a(xy)=b+c=\log_a x+\log_a y$$

The proof for the change of the base identity can be found in the last slide.

Relationship with Exponential Functions

The following two identities demonstrate that the operations of taking the base *b* logarithm and raising *b* to a power are ***inverse*** of each other.

Identity	Quick Examples
1. $\log_b(b^x) = x$	$\log_2(2^7) = 7$

The power to which you raise *b* in order to get b^x is *x*

2. $b^{\log b\ x} = x$	$5^{\log 5\ 8} = 8$

Raising *b* to the power to which it must be raised to get *x*, yields *x*

2.1.6 Solve Exponential and Logarithmic equations

Example 1

Solve the following equations

a. $5^{-x} = 125$ **b.** $3^{2x-1} = 6$

Example 1

Solve the following equations

a. $5^{-x} = 125$ **b.** $3^{2x-1} = 6$

a. Write the given equation $5^{-x} = 125$ in logarithmic form:

$$-x = \log_5 125$$

This gives

$$x = -\log_5 125 = -3$$

b. In logarithmic form, $3^{2x-1} = 6$ becomes

$$2x - 1 = \log_3 6$$

$$2x = 1 + \log_3 6$$

giving

$$\begin{aligned} x &= (1 + \log_3 6)/2 \\ &\approx (2.6309)/2 \\ &\approx 1.3155 \end{aligned}$$

Example 2

Solve the following equation

$$4^{x+1} = \frac{1}{3^{x-2}}$$

Example 2

Solve the following equation

$$4^{x+1} = \frac{1}{3^{x-2}}$$

Solution (1):

$$4^{x+1} = 3^{-(x-2)}$$

$$4^{x+1} = 3^{2-x}$$

$$\log_{10} 4^{x+1} = \log_{10} 3^{2-x}$$

$$(x+1)\log_{10} 4 = (2-x)\log_{10} 3$$

$$x\log_{10} 4 + \log_{10} 4 = 2\log_{10} 3 - x$$

Example 2

Solve the following equation

$$x(\log_{10} 4 + \log_{10} 3) = 2\log_{10} 3 - \log_{10} 4$$

$$x(\log_{10} 4 + \log_{10} 3) = 2\log_{10} 3 - \log_{10} 4$$

$$x = \frac{2\log_{10} 3 - \log_{10} 4}{\log_{10} 4 + \log_{10} 3}$$

$$x = \frac{\log_{10} \frac{9}{4}}{\log_{10} 12} \approx 0.33$$

Solution (2):

$$4^{x+1} = 3^{-(x-2)}$$

$$4^{x+1} = 3^{2-x}$$

$$\log_{10} 4^{x+1} = \log_{10} 3^{2-x}$$

$$\frac{\log_4 4^{x+1}}{\log_4 10} = \frac{\log_3 3^{2-x}}{\log_3 10}$$

$$(x+1)\log_3 10 = (2-x)\log_4 10$$

$$x\log_3 10 + \log_3 10 = 2\log_4 10 - x$$

$$x\log_3 10+\log_3 10=2\log_4 10-x\log_4 10$$

$$x(\log_3 10+\log_4 10)=2\log_4 10-\log_3 10$$

$$x=\frac{2\log_4 10-\log_3 10}{\log_3 10+\log_4 10}\approx 0.33$$

Change the base of a log

Change-of-Base Formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example 3

$$\log_{11} 9 = \frac{\log 9}{\log 11} \approx 0.91631$$

Your turn (Example 4)

Solve simultaneous equations, giving your answers as exact fractions:

$8^y = 4^{2x+3}$

$\log_2 y = \log_2 x + 4$

Your turn (Example 4)

Solve simultaneous equations, giving your answers as exact fractions:

$$8^y = 4^{2x+3}$$

$$\log_2 y = \log_2 x + 4$$

Solutions:

$$8^y = 4^{2x+3}$$

$$(2^3)^y = (2^2)^{2x+3}$$

$$2^{3y} = 2^{2(2x+3)}$$

$$3y = 4x + 6 \quad ①$$

$$\log_2 y - \log_2 x = 4$$

$$\log_2 \frac{y}{x} = 4$$

$$\frac{y}{x} = 2^4 = 16$$

$$y = 16x \quad ②$$

Substitute ② into

$\log_2 y - \log_2 x = 4$

$\log_2 \frac{y}{x} = 4$

$\frac{y}{x} = 2^4 = 16$

$y = 16x$

$$48x = 4x + 6$$

$$44x = 6$$

$$x = \frac{3}{22}$$

$$y = 16x = \frac{24}{11}$$

Your turn (Example 5)

If $xy = 64$ and $\log_x y + \log_y x = \frac{5}{2}$. Find x and y.

Your turn (Example 5)

If $xy = 64$ and $\log_x y + \log_y x = \frac{5}{2}$. Find x and y.

Solutions:

$$\log_x y + \log_y x = 5/2$$

$$\log_x y + \frac{1}{\log_x y} = 5/2$$

Let $\log_x y = u$

$$u + \frac{1}{u} = \frac{5}{2}$$

$$2u^2 + 2 = 5u$$

$$2u^2 - 5u + 2 = 0$$

$$(2u - 1)(u - 2) = 0$$

$u = \frac{1}{2}$ or $u = 2$

1) If $u = \frac{1}{2}$, $\log_x y = \frac{1}{2}$

$$\rightarrow y = x^{\frac{1}{2}} = \sqrt{x}$$

since $xy = 64$

$x\sqrt{x} = 64 \quad x^{\frac{3}{2}} = 64$

$x = 16 \quad y = \sqrt{x} = 4$

2) If $u = 2$, $\log_x y = 2$

$$\rightarrow y = x^2$$

since $xy = 64$

$$x^3 = 64$$

$x = 4 \quad y = x^2 = 16$

Your turn (Example 6)

a. Given that $3 + 2\log_2 x = \log_2 y$, show that $y = 8x^2$.

b. Hence, find the roots α and β, where α < β, of the equation

$$3 + 2\log_2 x = \log_2(14x - 3).$$

c. Show that $\log_2 \alpha = -2$.

d. Calculate $\log_2 \beta$, giving your answer to 3 significant figures.

Your turn (Example 6)

Solutions:

a. $3 + 2\log_2 x = \log_2 y$

$\log_2 y - 2\log_2 x = 3$

$\log_2 y - \log_2 x^2 = 3$

$\log_2 \frac{y}{x^2} = 3 \qquad \frac{y}{x^2} = 2^3 = 8$

$y = 8x^2$

b. Comparing equations,

$y = 14x - 3$

$8x^2 = 14x - 3$

$8x^2 - 14x + 3 = 0$

$(4x - 1)(2x - 3) = 0$

$x = 0.25$ or $1.5 \rightarrow \alpha = 0.25 \;\; \beta = 1.5$

c. $\log_2 \alpha = \log_2 \frac{1}{4} = -2$

since $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

d. $\log_2 \beta = \log_2 \frac{3}{2}$

$$\log_2 1.5 = \frac{\log_{10} 1.5}{\log_{10} 2} = 0.585\ (3\ s.f.)$$

Learning outcomes

At the end of this lecture, you should be able to;

2.1.1 **Sketch** the **graph** of **Exponential function**

2.1.2 **Write** an **expression** in **logarithmic form**

2.1.3 **Recognize** that the **Logarithmic function** is an **inverse** of **Exponential function**

2.1.4 **Sketch** the **graph** of **Logarithmic function**

2.1.5 **Apply Laws** of **logarithms**

2.1.6 **Solve Exponential** and **Logarithmic equations**

Formulas to memorize

Laws of Logarithms:

1. $\log_b(xy) = \log_b x + \log_b y$

2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

3. $\log_b(x^r) = r \log_b x$

4. $\log_b b = 1;\ \log_b 1 = 0$

5. $\log_b\left(\dfrac{1}{x}\right) = -\log_b x$

6. $\log_b x = \dfrac{\log_a x}{\log_a b}$

Preview activity: Modelling with Exponential and Logarithmic functions

Watch this video

https://www.youtube.com/watch?v=0BSaMH4hINY

Preview activity: Modelling with Exponential and Logarithmic functions

How do you think…

1. Which nature events can be modelled by using Exponential functions?
2. Can we use only Natural Exponential function for the modelling instead of using Exponential functions with different bases?
3. Which nature events can be modelled by using Logarithmic functions?

Change the base of a log

Proof of the Change-of-Base Formula

From

$\log_a x = m$

we obtain

$a^m = x$

and therefore

$\log_b(a^m) = \log_b(x)$

then

$m \log_b a = \log_b x$

and finally

$$m = \frac{\log_b x}{\log_b a} \quad \Rightarrow \quad \log_a x = \frac{\log_b x}{\log_b a}$$