CSE 326: Data Structures
Lecture #22

Mergeable Heaps

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Winter Quarter 2002

Summary of Heap ADT Analysis

- Consider a heap of N nodes
- Space needed: O(N)
  - Actually, O(MaxSize) where MaxSize is the size of the array
  - Pointer-based implementation: pointers for children and parent
    - Total space = 3N + 1 (3 pointers per node + 1 for size)
- FindMin: O(1) time; DeleteMin and Insert: O(log N) time
- BuildHeap from N inputs: What is the run time?
  - N Insert operations = O(N log N)
  - O(N): Treat input array as a heap and fix it using percolate down
    - *Thanks, Floyd!*

Other Heap Operations

- Find and FindMax: O(N)
- DecreaseKey(P,Δ,H): Subtract Δ from current key value at position P and percolate up. Running Time: O(log N)
- IncreaseKey(P,Δ,H): Add Δ to current key value at P and percolate down. Running Time: O(log N)
  - *E.g.* Schedulers in OS often decrease priority of CPU-hogging jobs
- Delete(P,H): Use DecreaseKey (to 0) followed by DeleteMin. Running Time: O(log N)
  - *E.g.* Delete a job waiting in queue that has been preemptively terminated by user

But What About...

- Merge(H1,H2): Merge two heaps H1 and H2 of size O(N).
  - *E.g.* Combine queues from two different sources to run on one CPU.

1. Can do O(N) Insert operations: O(N log N) time
2. Better: Copy H2 at the end of H1 (assuming array implementation) and use Floyd's Method for BuildHeap.

   Running Time: O(N)

Can we do even better? (*i.e.* Merge in O(log N) time?)

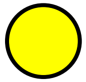
Binomial Queues

- Binomial queues support all three priority queue operations Merge, Insert and DeleteMin in O(log N) time
- Idea: Maintain a collection of heap-ordered trees
  - *Forest of binomial trees*
- Recursive Definition of Binomial Tree (based on height k):
  - Only one binomial tree for a given height
  - Binomial tree of height 0 = single root node
  - Binomial tree of height k = B_k = Attach B_{k-1} to root of another B_{k-1}

Building a Binomial Tree

- To construct a binomial tree B_k of height k:
  1. Take the binomial tree B_{k-1} of height k-1
  2. Place another copy of B_{k-1} one level below the first
  3. Attach the root nodes
- Binomial tree of height k has exactly 2^k nodes (by induction)

B_0 B_1 B_2 B_3



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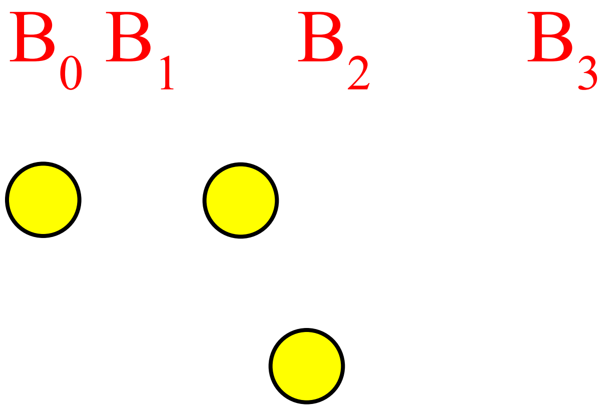

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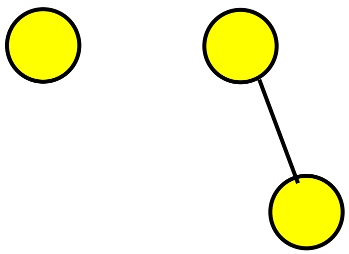

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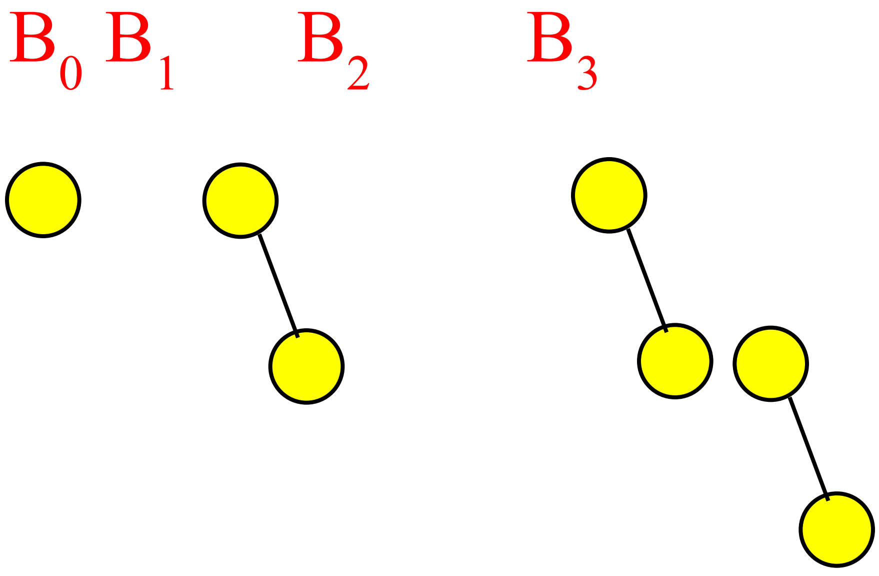

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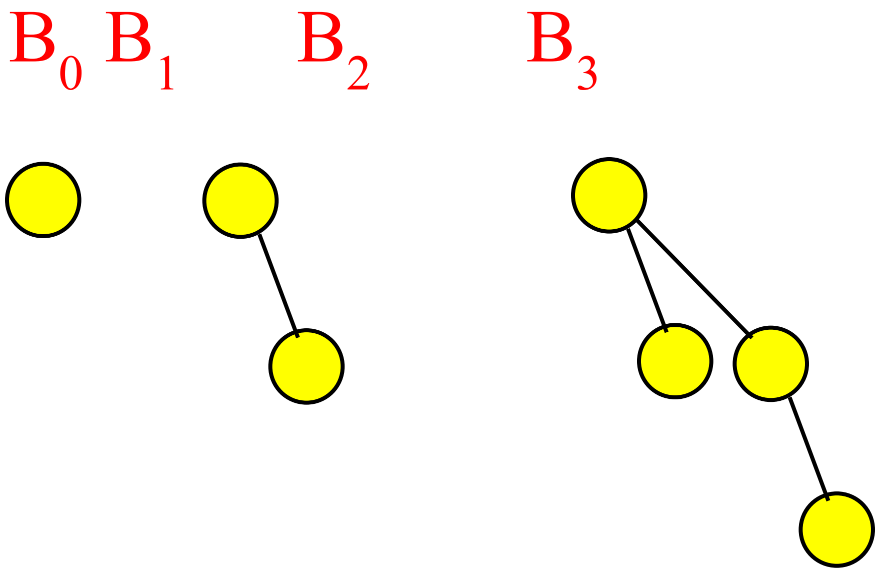

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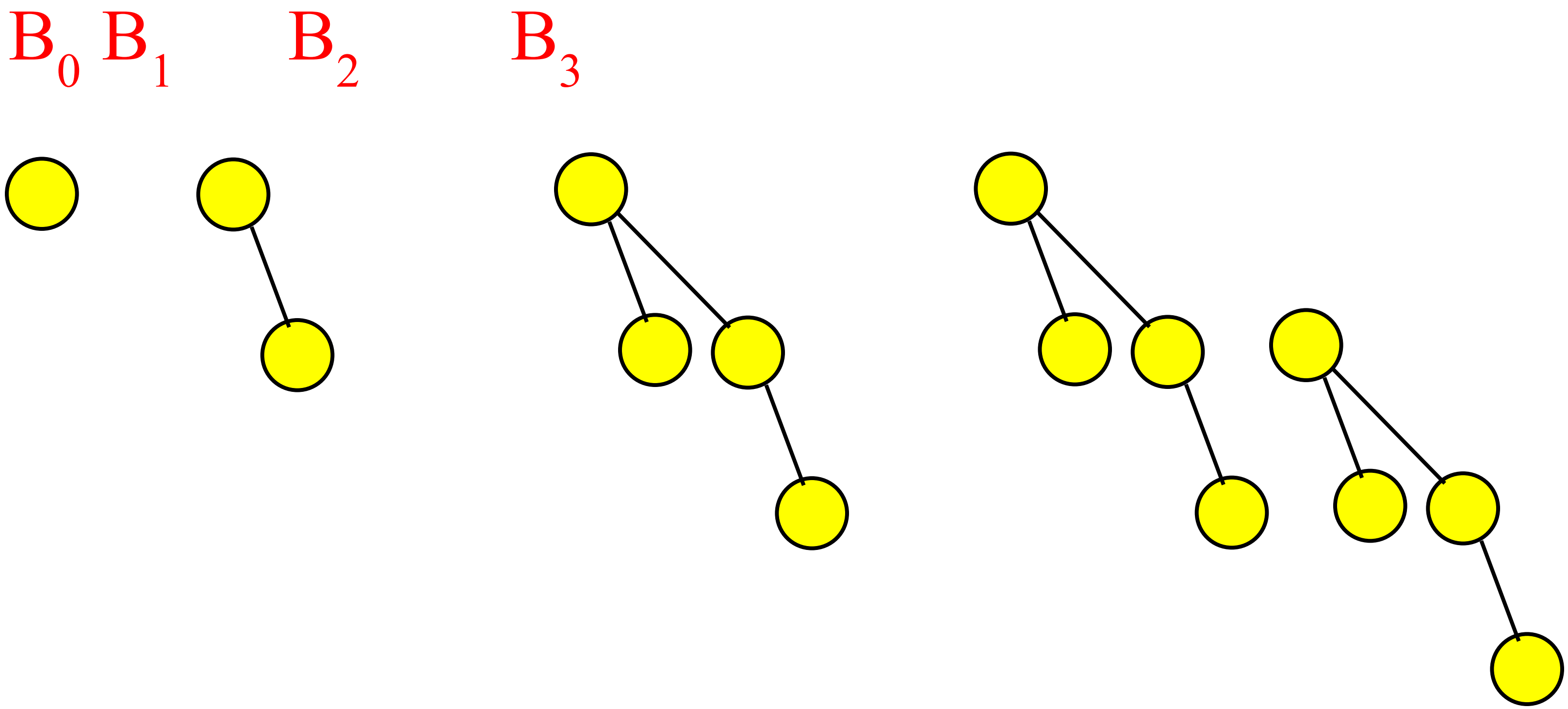

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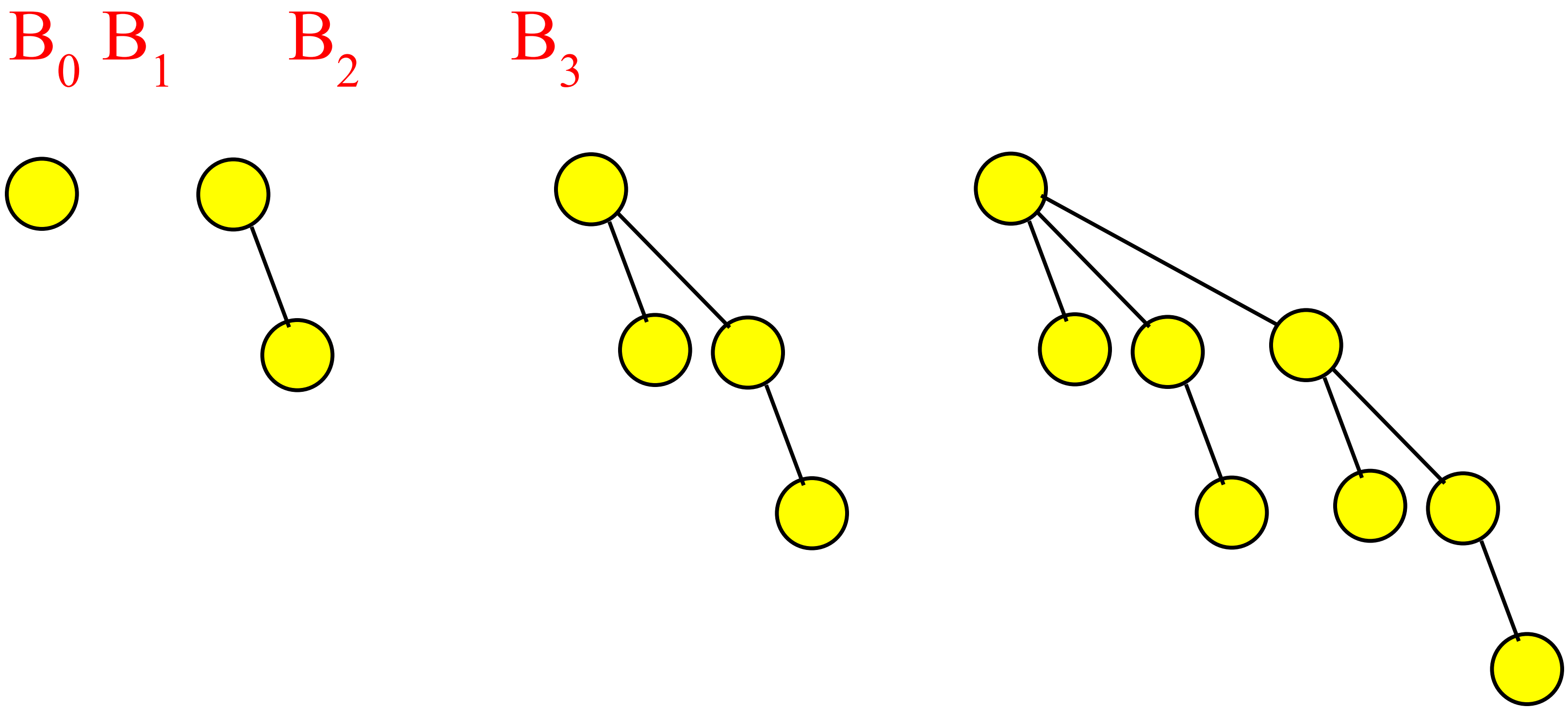
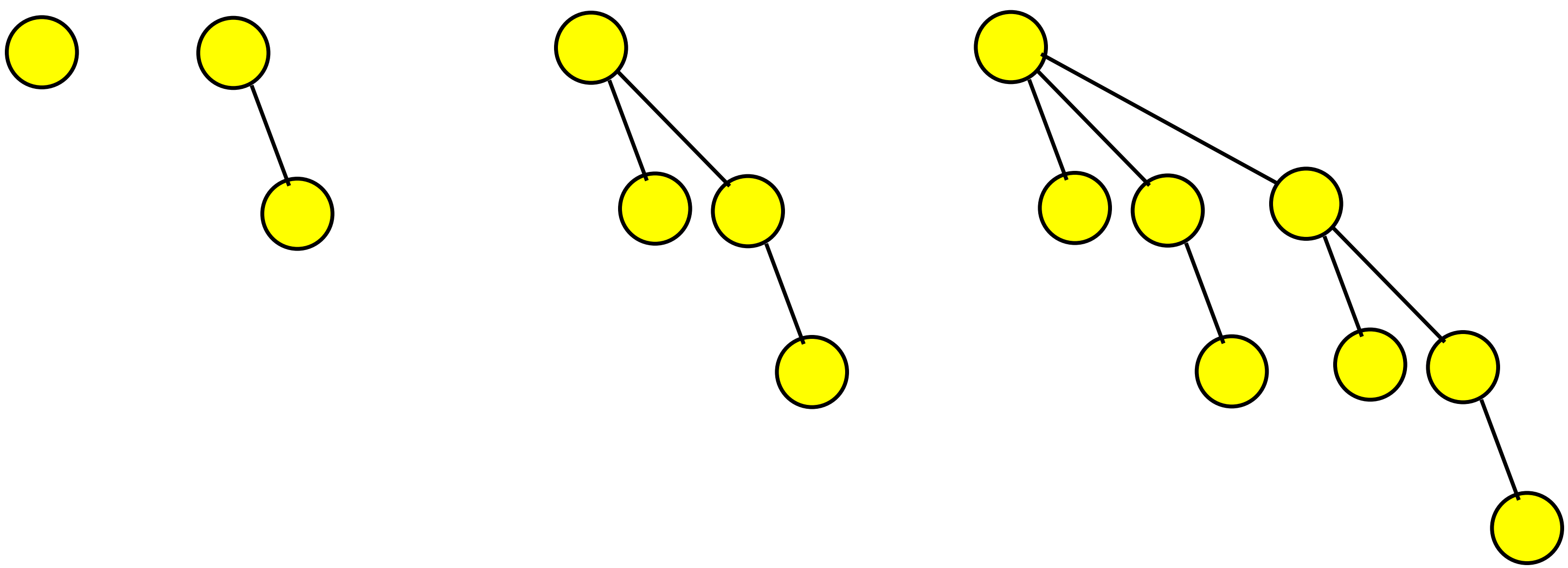


Why Binomial?

- Why are these trees called binomial?
  - Hint: how many nodes at depth d?

B_0 B_1 B_2 B_3



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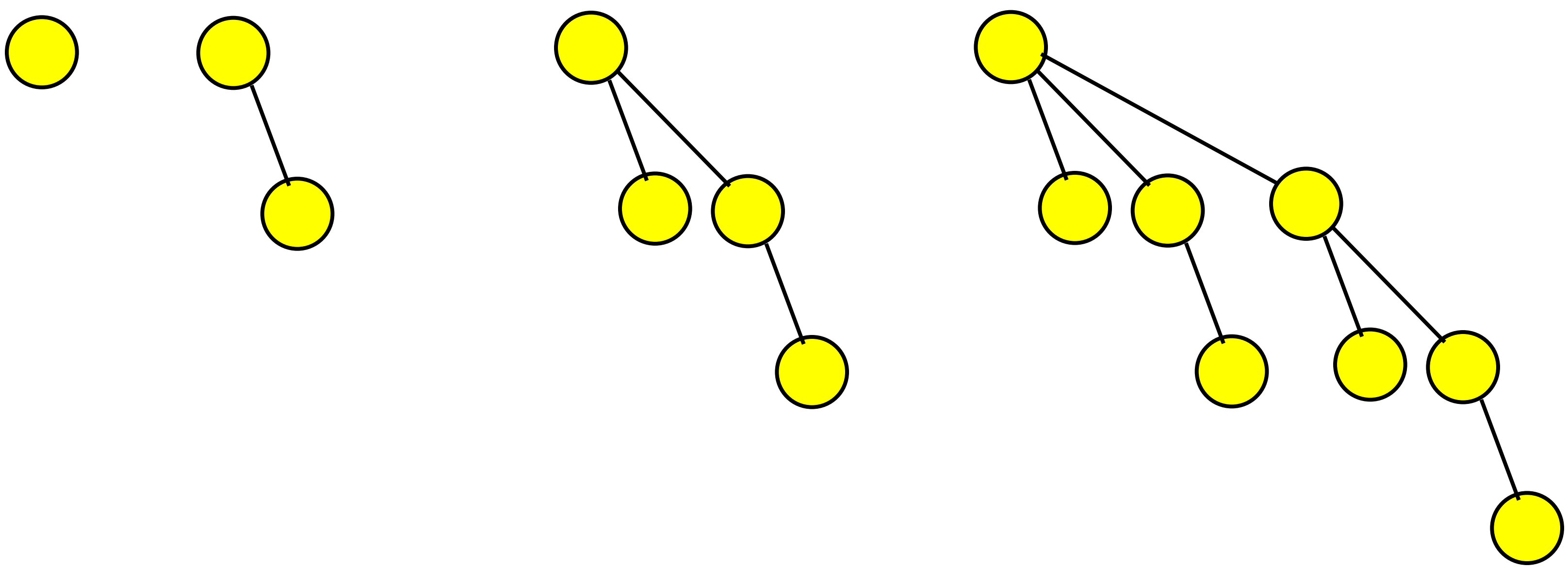
Number of nodes at different depths d for B_k =
[1], [1 1], [1 2 1], [1 3 3 1], …

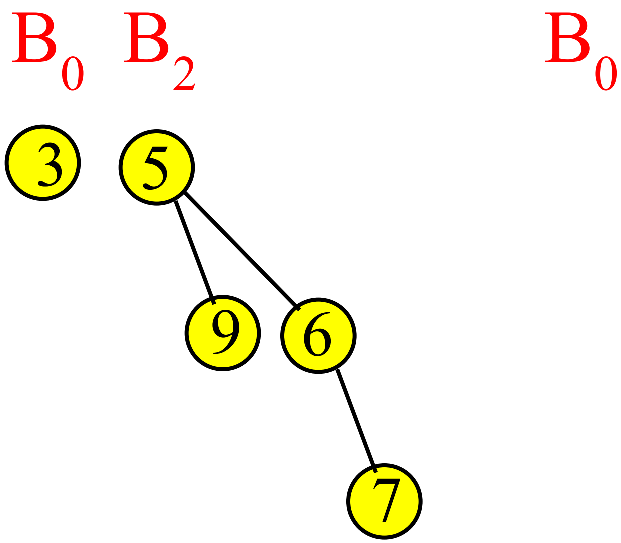
Binomial coefficients of $(a + b)^k = k!/((k-d)!d!)$

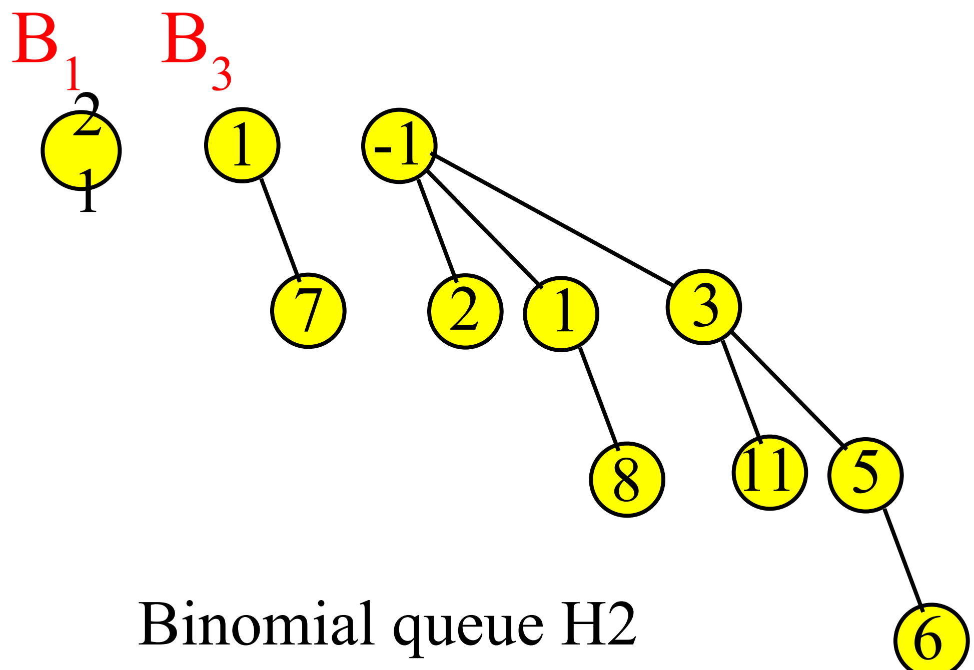
B_0 B_1 B_2 B_3



Definition of Binomial Queues

Binomial Queue = “forest” of heap-ordered binomial trees





Binomial queue H1
5 elements = 101 base 2
□ $B_2 B_0$

Binomial queue H2
11 elements = 1011 base 2
□ $B_3 B_1 B_0$

Binomial Queue Properties

Suppose you are given a binomial queue of N nodes

1. There is a unique set of binomial trees for N nodes
2. What is the maximum number of trees that can be in an N-node queue?
   - 1 node 1 tree B_0; 2 nodes 1 tree B_1; 3 nodes 2 trees B_0 and B_1; 7 nodes 3 trees B_0, B_1 and B_2 …
   - Trees B_0, B_1, …, B_k can store up to $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$ nodes = N.
   - Maximum is when all trees are used. So, solve for (k+1).
   - Number of trees is $\leq \log(N+1) = O(\log N)$

Binomial Queues: Merge

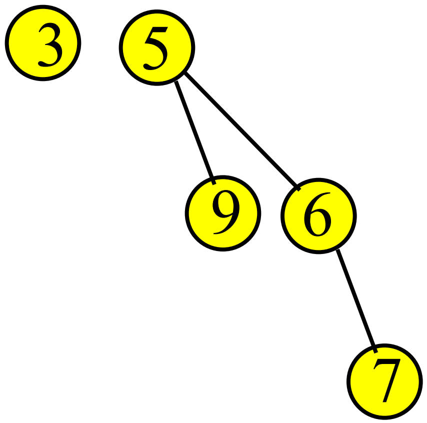
- Main Idea: Merge two binomial queues by merging individual binomial trees
  - Since B_{k+1} is just two B_k's attached together, merging trees is easy
- Steps for creating new queue by merging:

1. Start with B_k for smallest k in either queue.
2. If only one B_k, add B_k to new queue and go to next k.
3. Merge two B_k's to get new B_{k+1} by making larger root the child of smaller root. Go to step 2 with k = k + 1.

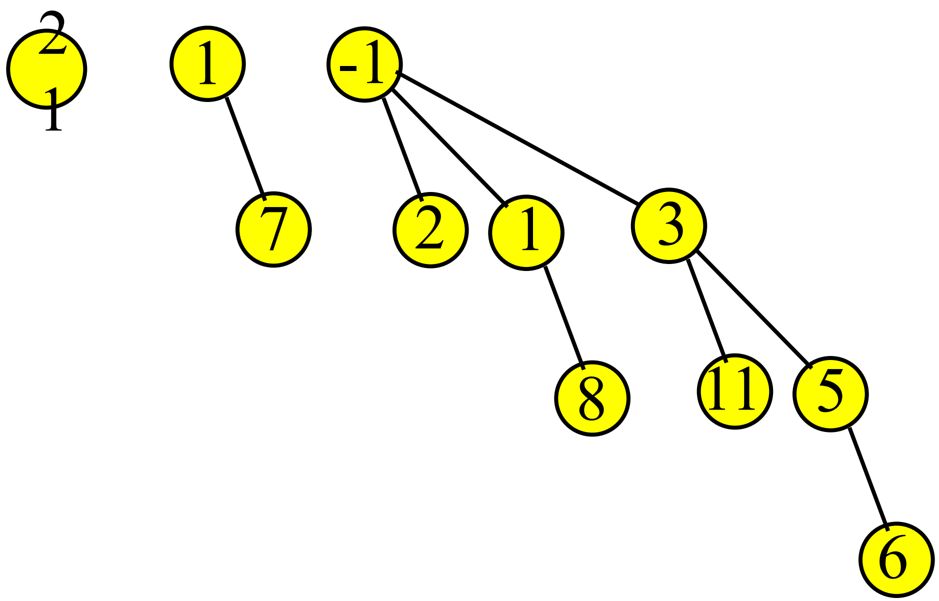
Example: Binomial Queue Merge

- Merge H1 and H2

H1:



H2:



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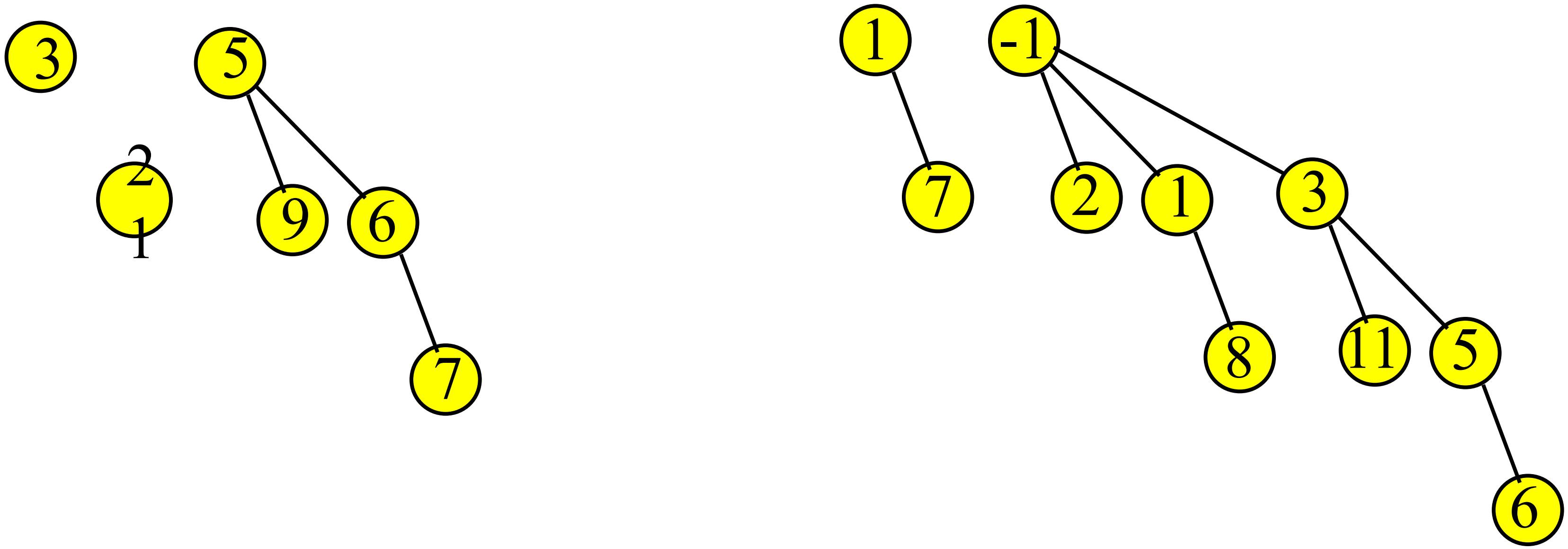
H1:

H2:



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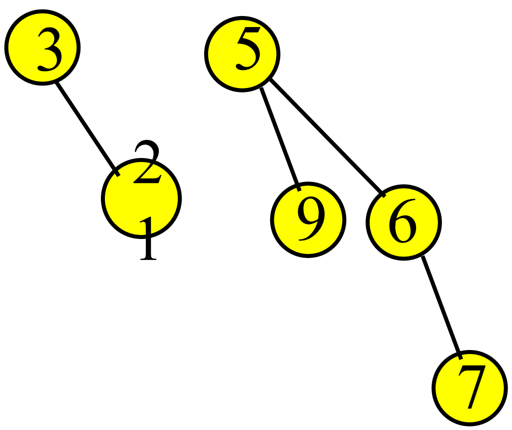
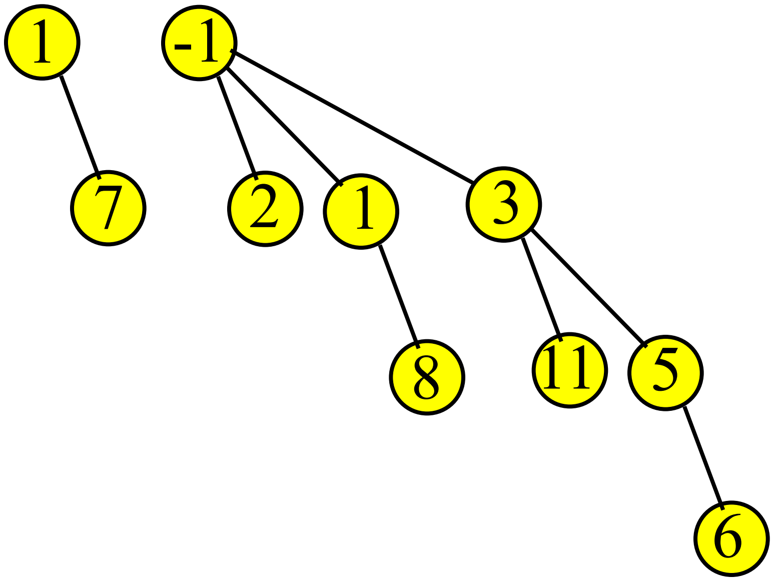


H2:



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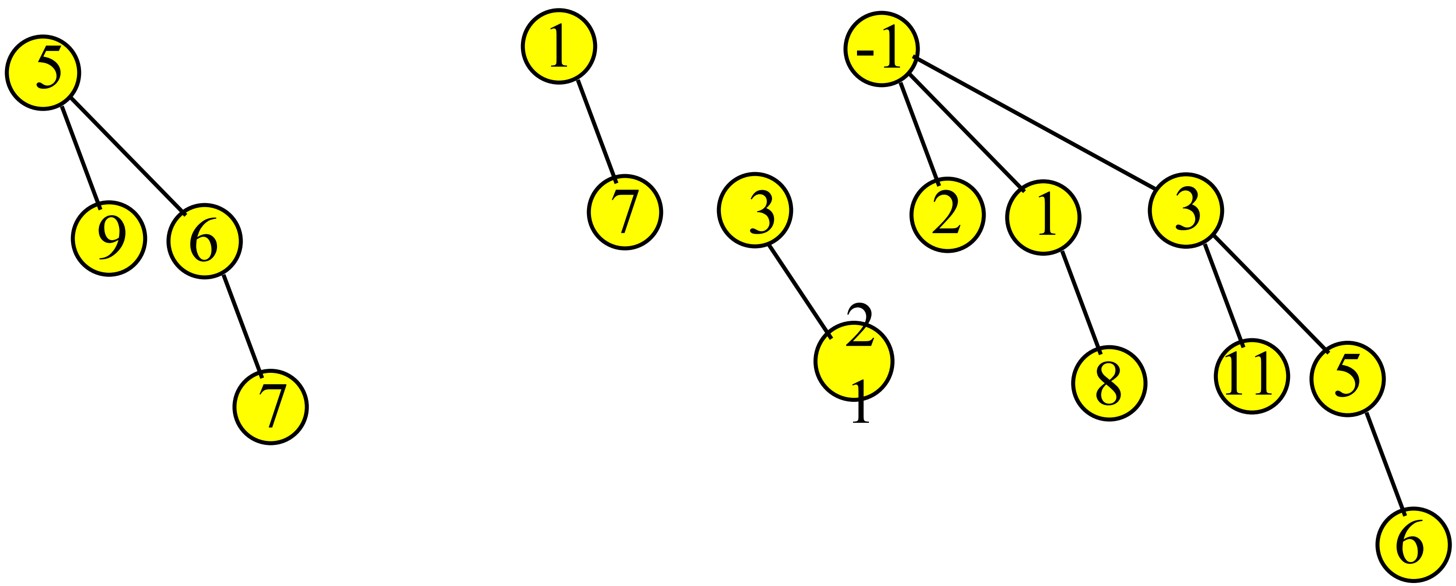
H1: H2:



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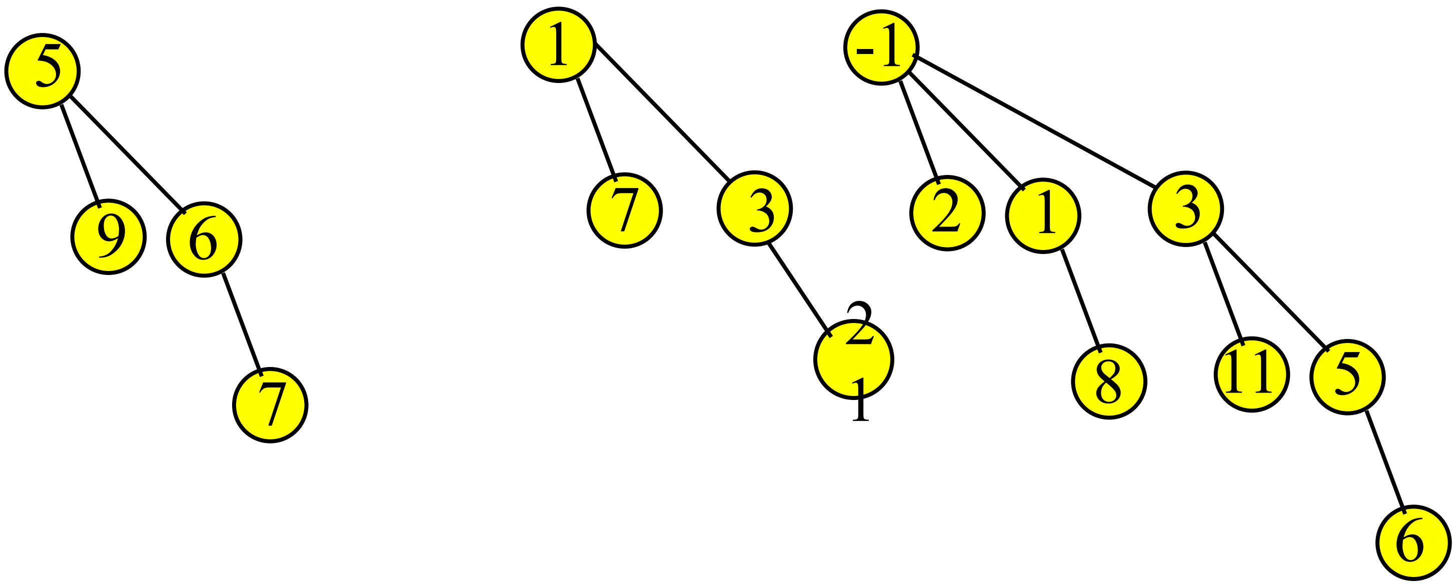

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- Merge H1 and H2

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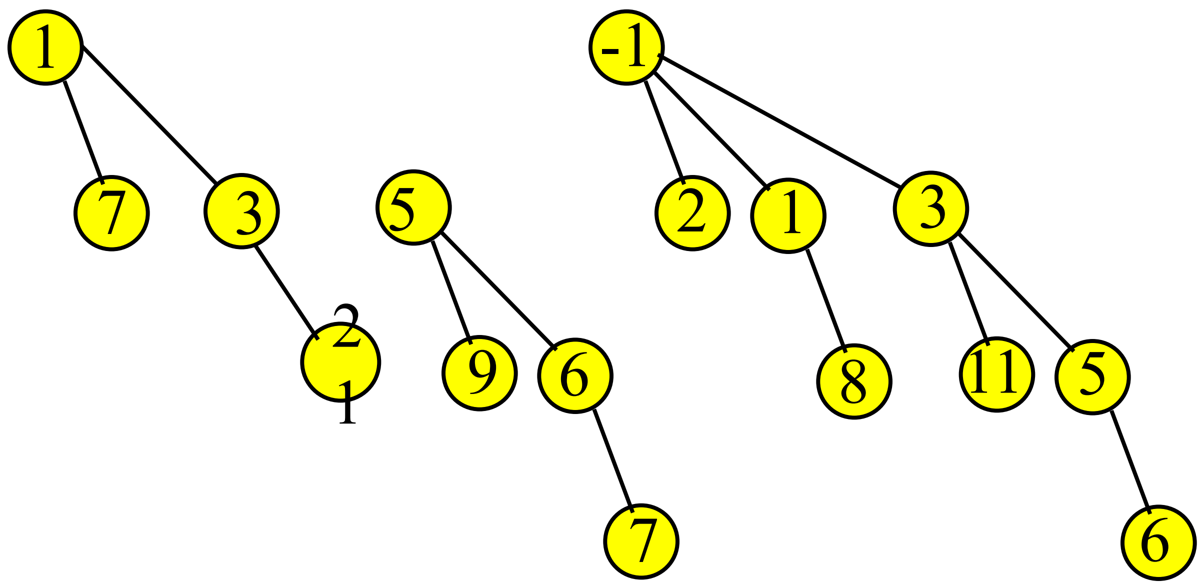

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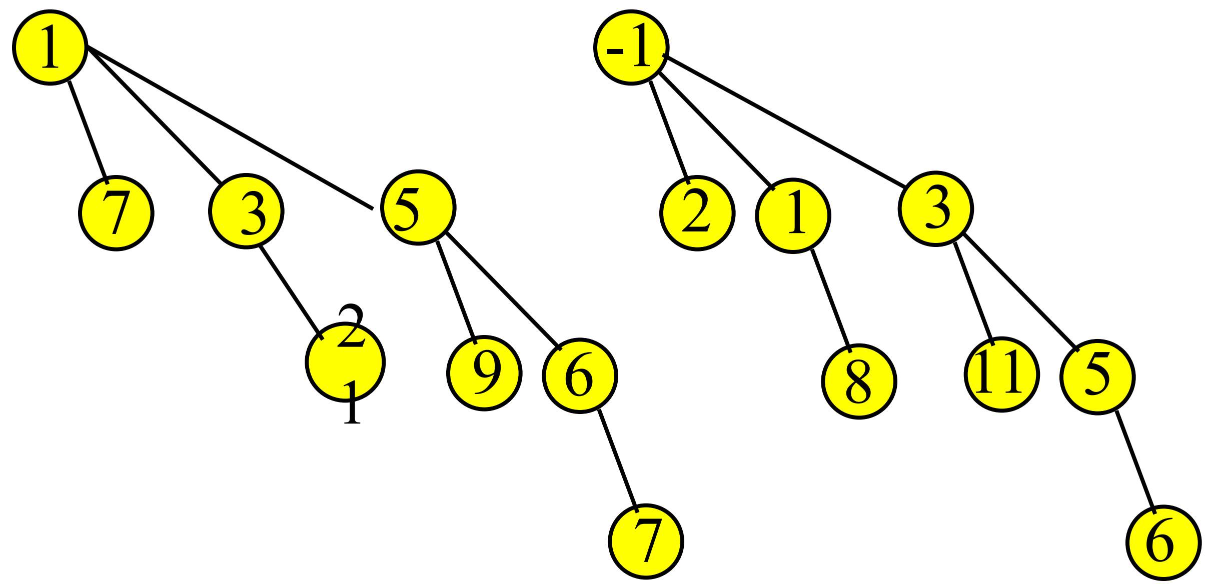

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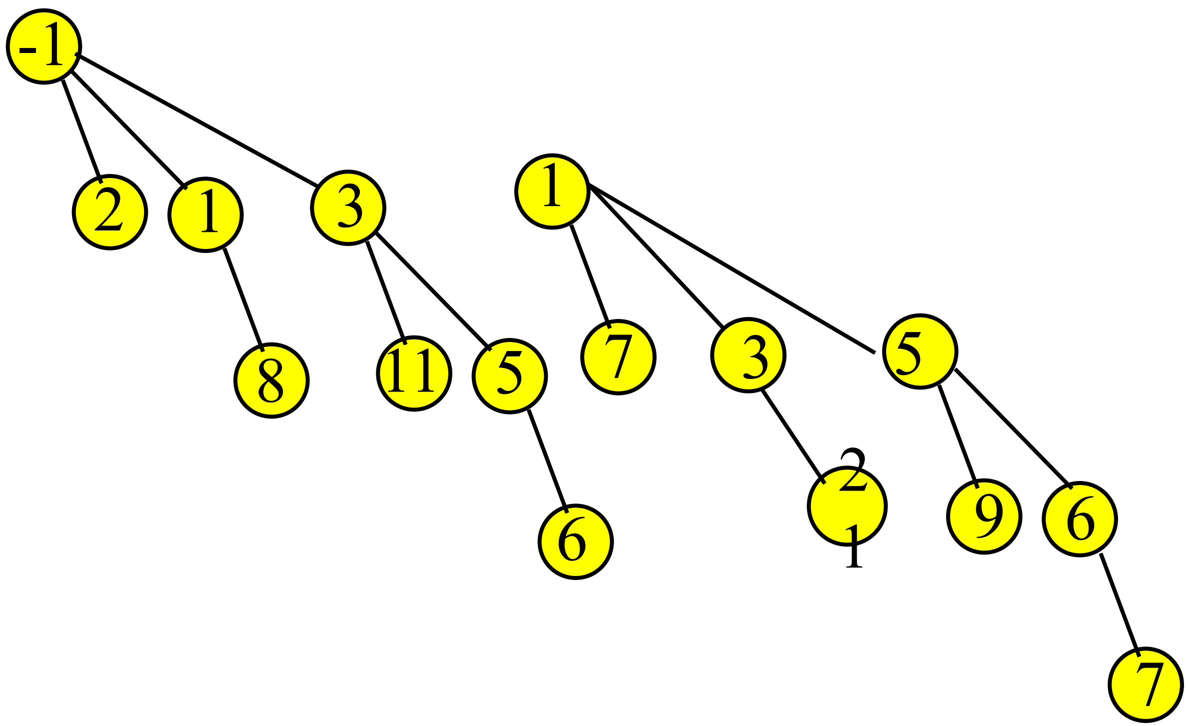

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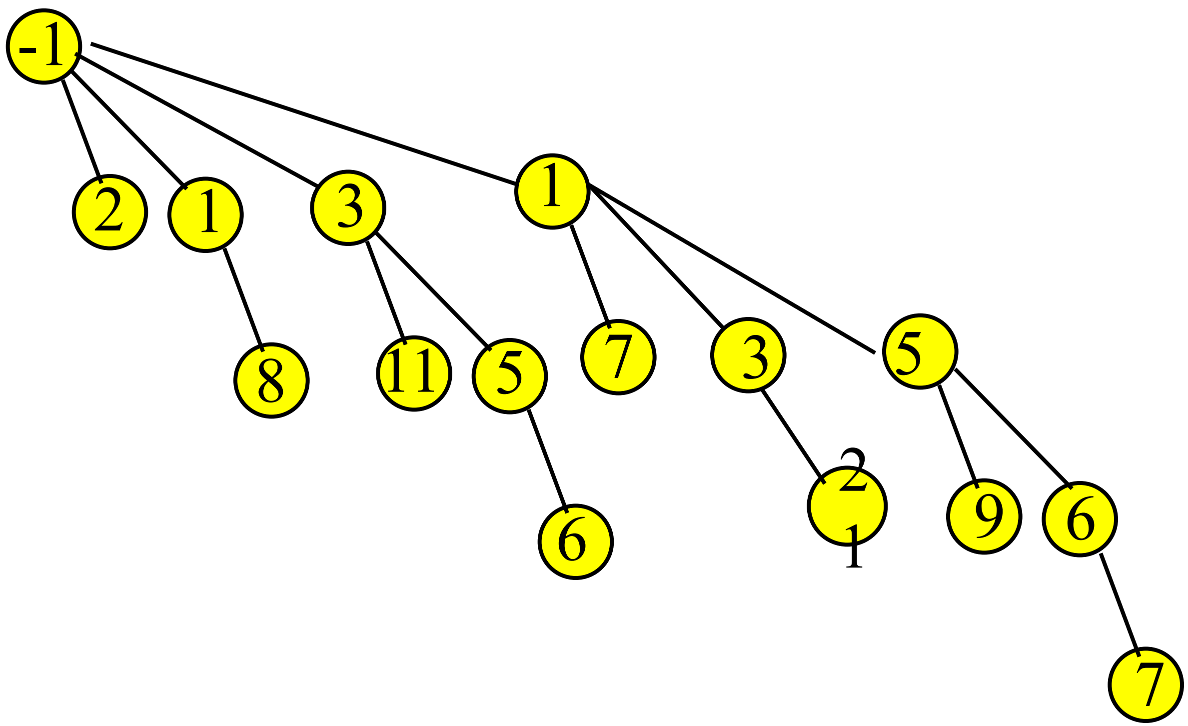

Binomial Queues: Merge and Insert

- What is the run time for Merge of two O(N) queues?
- How would you insert a new item into the queue?

Binomial Queues: Merge and Insert

- What is the run time for Merge of two O(N) queues?
  - O(number of trees) = O(log N)
- How would you insert a new item into the queue?
  - Create a single node queue B_0 with new item and merge with existing queue
  - Again, O(log N) time
- Example: Insert 1, 2, 3, …,7 into an empty binomial queue

Insert 1,2,…,7

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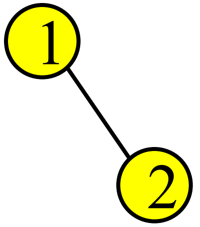

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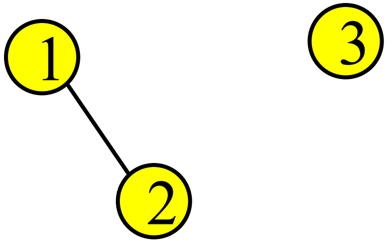

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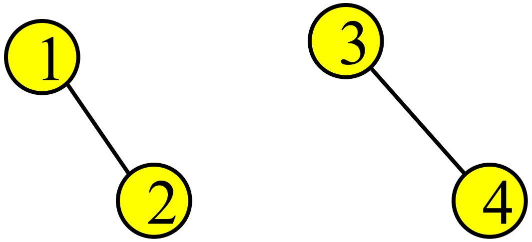

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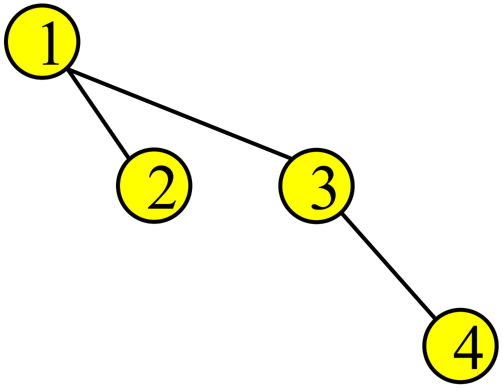

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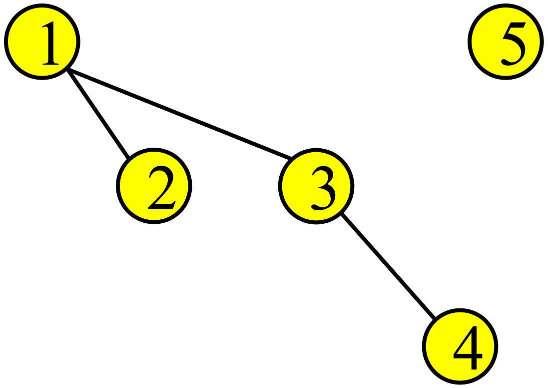

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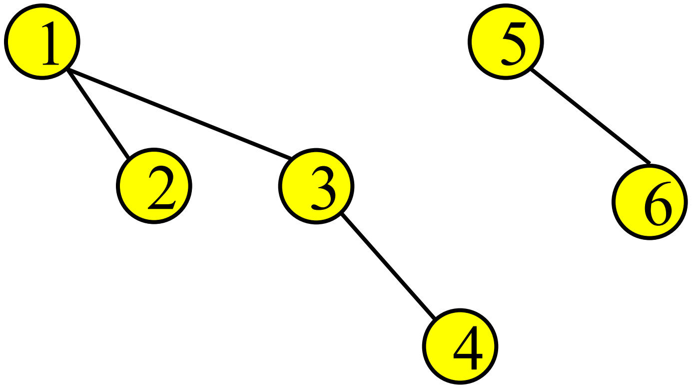

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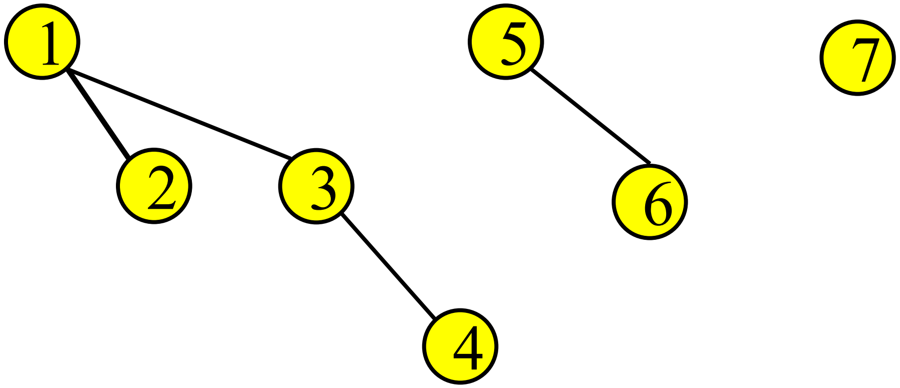
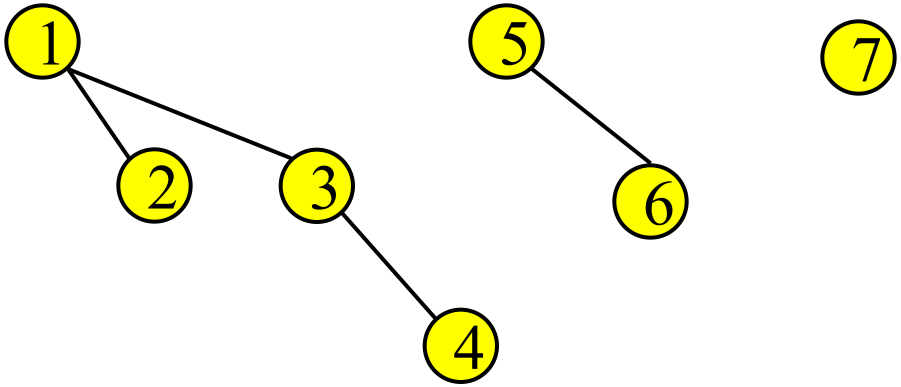


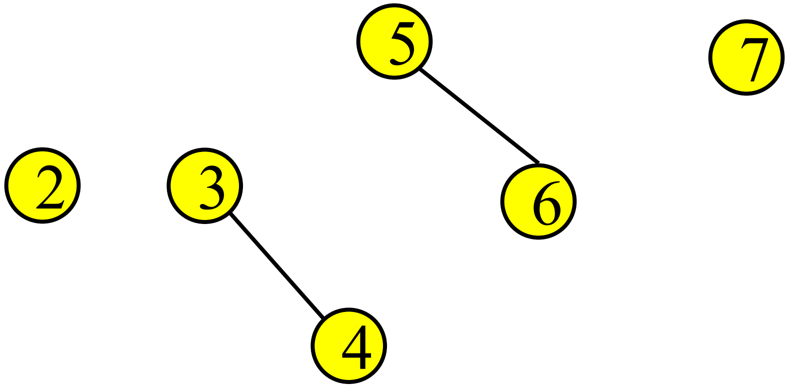
Binomial Queues: DeleteMin

- Steps:
  1. Find tree B_k with the smallest root
  2. Remove B_k from the queue
  3. Delete root of B_k (return this value); You now have a new queue made up of the forest B_0, B_1, …, B_{k-1}
  4. Merge this queue with remainder of the original (from step 2)
- Run time analysis: Step 1 is O(log N), step 2 and 3 are O(1), and step 4 is O(log N). Total time = O(log N)
- Example: Insert 1, 2, …, 7 into empty queue and DeleteMin

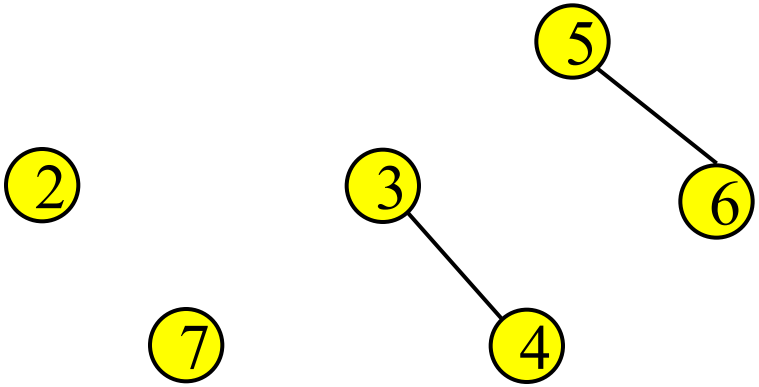
Insert 1,2,…,7



DeleteMin



Merge



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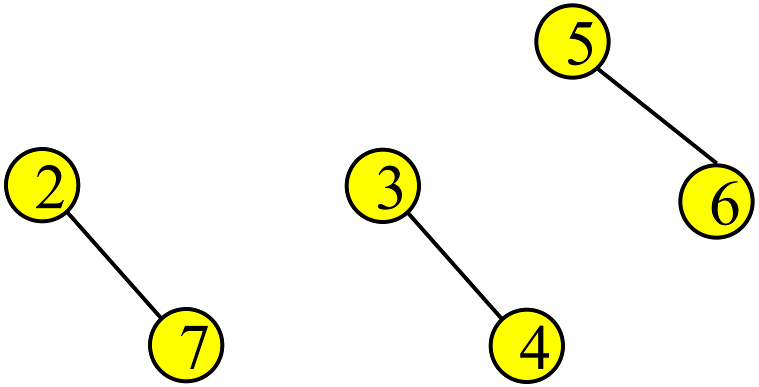

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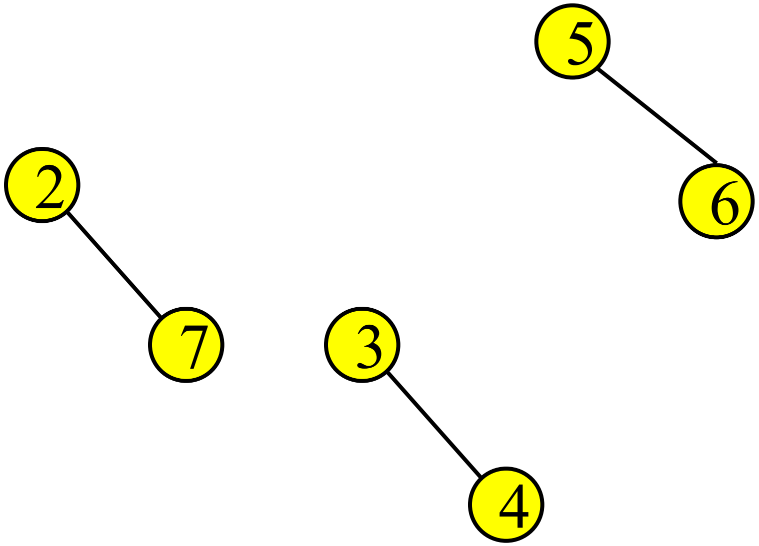

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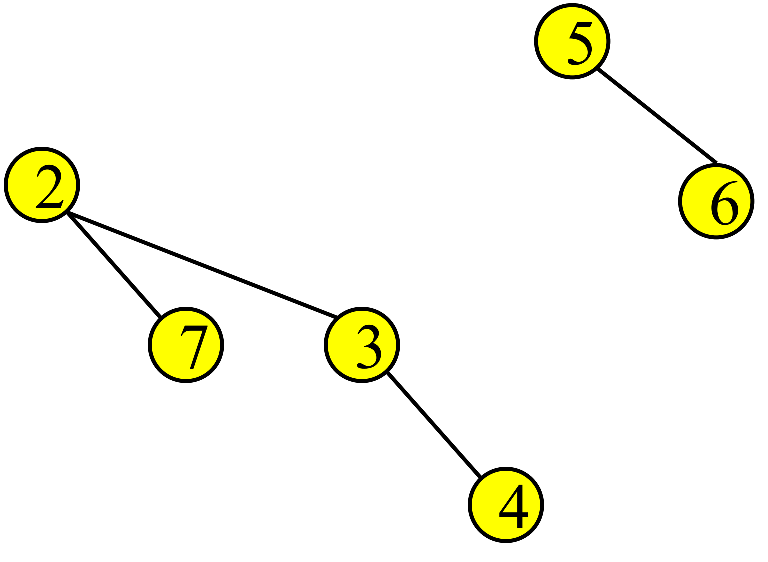

DONE!

Implementation of Binomial Queues

- Need to be able to scan through all trees, and given two binomial queues find trees that are same size
  - Use array of pointers to root nodes, sorted by size
  - Since is only of length log(N), don't have to worry about cost of copying this array
  - At each node, keep track of the size of the (sub) tree rooted at that node
- Want to merge by just setting pointers
  - Need pointer-based implementation of heaps
- DeleteMin requires fast access to all subtrees of root
  - Use First-Child/Next-Sibling representation of trees

Other Mergeable Priority Queues: Leftist and Skew Heaps

- Leftist Heaps: Binary heap-ordered trees with left subtrees always "longer" than right subtrees
  - Main idea: Recursively work on right path for Merge/Insert/DeleteMin
  - Right path is always short has O(log N) nodes
  - Merge, Insert, DeleteMin all have O(log N) running time (see text)
- Skew Heaps: Self-adjusting version of leftist heaps (*a la* splay trees)
  - Do not actually keep track of path lengths
  - Adjust tree by swapping children during each merge
  - O(log N) amortized time per operation for a sequence of M operations
- We will skip details… just recognize the names as mergeable heaps!

Coming Up

- Some random randomized data structures
  - Treaps
  - Skip Lists
  - FOR MONDAY: Read section on randomized data structures in Weiss. Be prepared, if called on, to explain in your own words why we might want to use a data structure that incorporates randomness!