Method of moments
for thin wires

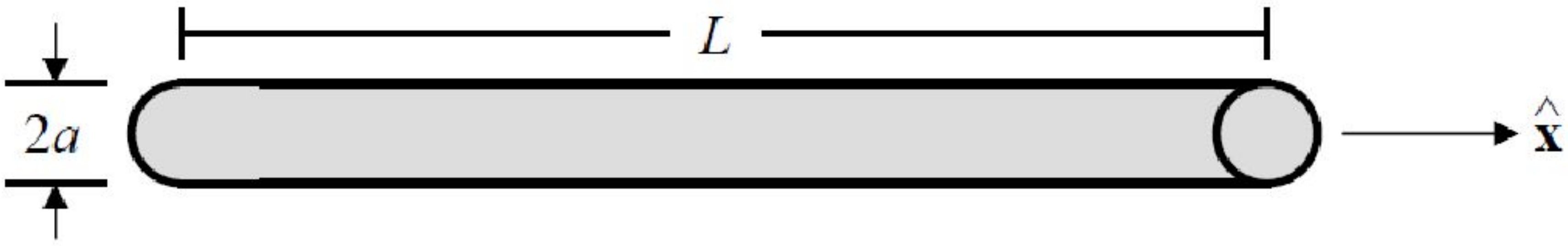


Figure 3.1: Thin wire dimensions.

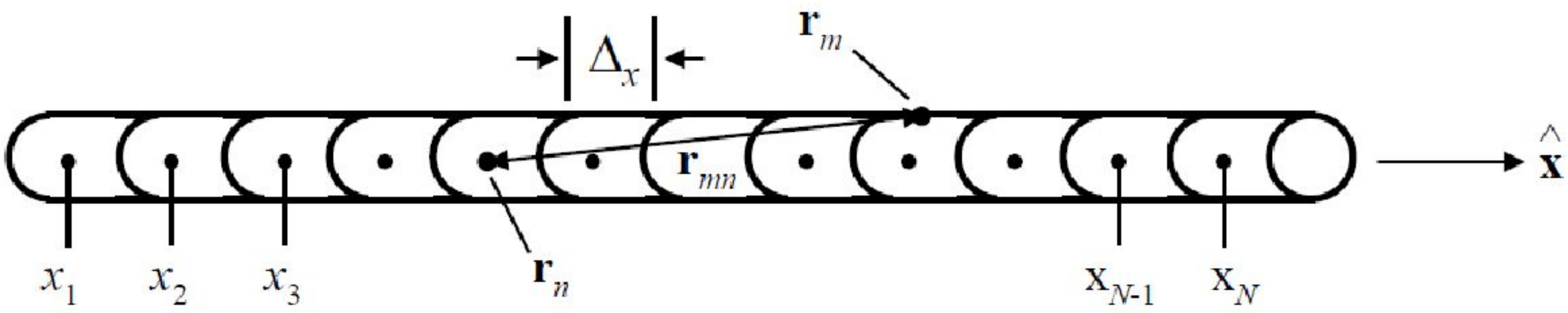


Figure 3.2: Thin wire segmentation.

The Method of Moments

$$\mathbf{L}\mathbf{f} = \mathbf{g}$$

$$\mathbf{f} = \sum_n a_n \mathbf{v}_n \qquad \sum_n a_n \mathbf{L}\mathbf{v}_n = \mathbf{g}$$

$$\sum_n a_n \langle \mathbf{v}_m, \mathbf{L}\mathbf{v}_n \rangle = \langle \mathbf{v}_m, \mathbf{g} \rangle$$

$$\begin{bmatrix} \langle \mathbf{v}_1, \mathbf{L}\mathbf{v}_1 \rangle & \langle \mathbf{v}_1, \mathbf{L}\mathbf{v}_2 \rangle & \\ \langle \mathbf{v}_2, \mathbf{L}\mathbf{v}_1 \rangle & \langle \mathbf{v}_2, \mathbf{L}\mathbf{v}_2 \rangle & \\ & & \ddots \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \langle \mathbf{v}_1, \mathbf{L}\mathbf{g} \rangle \\ \langle \mathbf{v}_2, \mathbf{L}\mathbf{g} \rangle \\ \vdots \\ \langle \mathbf{v}_N, \mathbf{L}\mathbf{g} \rangle \end{bmatrix}$$

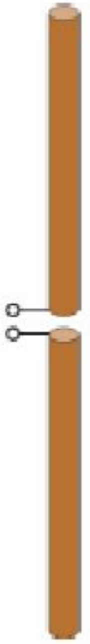
Galerkin Method

- Converts a linear equation to a matrix equation

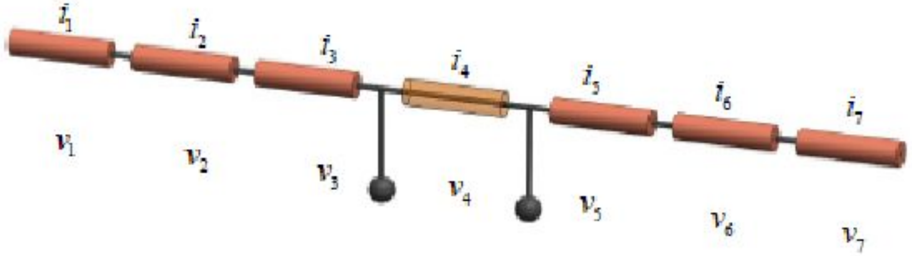
Integral Equation

- Usually uses PEC approximation
- Usually based on current

$$E_z^{\text{inc}} = \frac{j}{\omega\varepsilon} \int_{-L/2}^{L/2} I_z(z') \left(k^2 + \frac{\partial^2}{\partial z^2} \right) \frac{e^{-jkr}}{4\pi r} dz'$$



The Method of Moments



$$\begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} & z_{17} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} & z_{27} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} & z_{37} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} & z_{46} & z_{47} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} & z_{56} & z_{57} \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} & z_{67} \\ z_{71} & z_{72} & z_{73} & z_{74} & z_{75} & z_{76} & z_{77} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}$$

$$A_z(\rho, z) = \mu \int_{-L/2}^{L/2} I_z(z') \frac{e^{-jkr}}{4\pi r}\, dz' \tag{4.8}$$

$$r = \sqrt{(z - z')^2 + \rho^2} \tag{4.7}$$

$$E_z^s = -j\omega A_z - \frac{j}{\omega\mu\epsilon} \frac{\partial^2}{\partial z^2} A_z \tag{4.9}$$

$$E_z^i = \frac{j}{\omega\mu\epsilon} \left[\frac{\partial^2}{\partial z^2} + k^2 \right] A_z \tag{4.10}$$

$$E_z^i(z) = \frac{j}{\omega\mu\epsilon}\left[\frac{\partial^2}{\partial z^2} + k^2\right] A_z = \frac{j}{\omega\epsilon}\left[\frac{\partial^2}{\partial z^2} + k^2\right] \int_{-L/2}^{L/2} I_z(z') \frac{e^{-jkr}}{4\pi r}\, dz' \quad (4.11)$$

which is called *Hallén's integral equation*

$$E_z^i(z) = \frac{j}{\omega\epsilon} \int_{-L/2}^{L/2} I_z(z') \left[\frac{\partial^2}{\partial z^2} + k^2\right] \frac{e^{-jkr}}{4\pi r}\, dz' \quad (4.12)$$

which is called *Pocklington's integral equation*

4.4 SOLVING POCKLINGTON'S EQUATION

Pocklington's equation,

$$-j\omega\epsilon E_z^i(z) = \int_{-L/2}^{L/2} I_z(z')\left[\frac{\partial^2}{\partial z^2} + k^2\right]\frac{e^{-jkr}}{4\pi r}\,dz' \tag{4.57}$$

can be solved by a straightforward application of the moment method, since the differential operator is inside the integral and acts on the Green's function only. Expanding the current into a sum of N weighted basis functions and applying N testing functions we obtain a linear system with matrix elements

$$z_{mn} = \int_{f_m} f_m(z) \int_{f_n} f_n(z')\left[\frac{\partial^2}{\partial z^2} + k^2\right]\frac{e^{-jkr}}{4\pi r}\,dz'\,dz \tag{4.58}$$

and excitation vector elements b_m given by

$$b_m = -j\omega\epsilon \int_{f_m} f_m(z)\,E_z^i(z)\,dz \tag{4.59}$$

$$L(f) = g \tag{3.24}$$

where L is a linear operator, g is a known forcing function, and f is unknown. In electromagnetic problems, L is typically an integro-differential operator, f is the unknown function (charge, current) and g is a known excitation source (incident field). Let us now expand f into a sum of N weighted *basis functions*,

$$f = \sum_{n=1}^{N} a_n f_n \tag{3.25}$$

where a_n are unknown weighting coefficients. Because L is linear, substitution of the above into (3.24) yields

$$\sum_{n=1}^{N} a_n L(f_n) \approx g \tag{3.26}$$

$$< f_m, f_n > = \int_{f_m} f_m(\mathbf{r}) \cdot \int_{f_n} f_n(\mathbf{r}') \, d\mathbf{r}' d\mathbf{r} \tag{3.28}$$

$$\sum_{n=1}^{N} a_n < f_m, L(f_n) > = < f_m, g > \tag{3.29}$$

which results in the $N \times N$ matrix equation $\mathbf{Za} = \mathbf{b}$ with matrix elements

$$z_{mn} = < f_m, L(f_n) > \tag{3.30}$$

and right-hand side vector elements

$$b_m = < f_m, g > \tag{3.31}$$

3.3.1 Pulse Functions

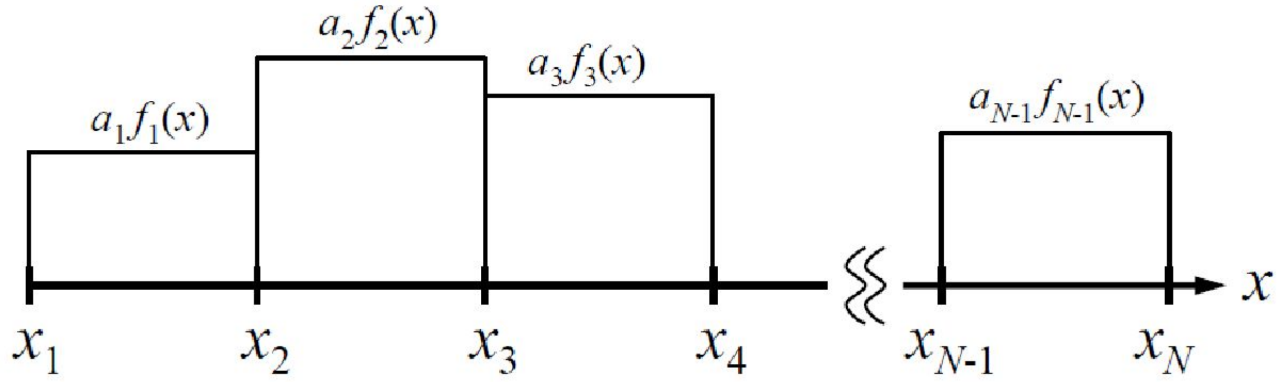


Figure 3.8: Pulse functions.

$$f_n(x) = 1 \qquad x_n \leq x \leq x_{n+1} \tag{3.33}$$

$$f_n(x) = 0 \qquad \text{elsewhere} \tag{3.34}$$

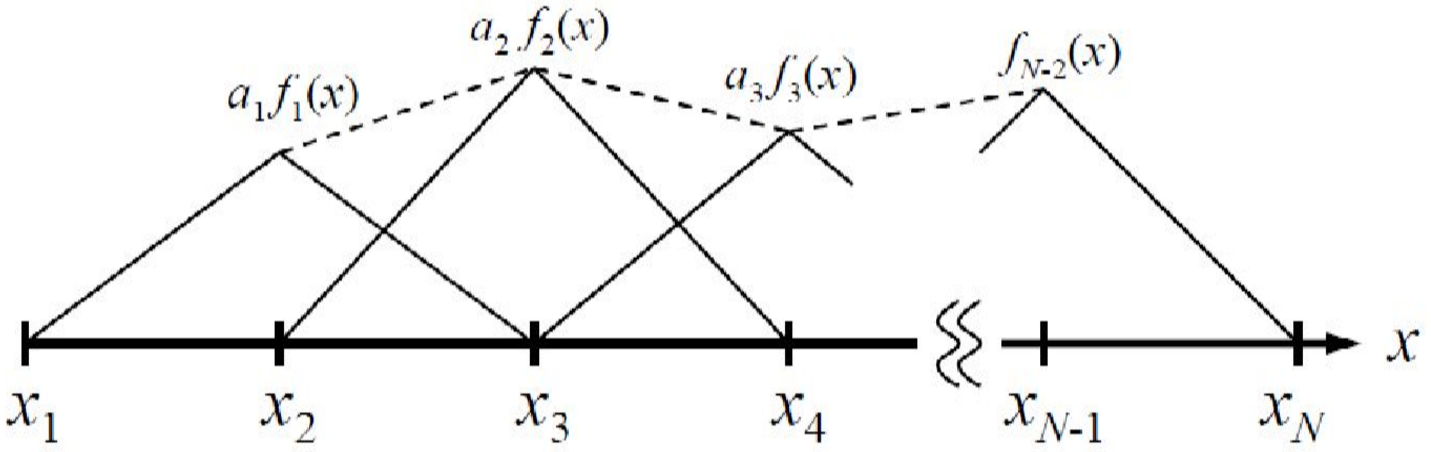


Figure 3.9: Triangle functions (end condition 1).

$$f_n(x) = \frac{x - x_{n-1}}{x_n - x_{n-1}} \qquad x_{n-1} \leq x \leq x_n \tag{3.35}$$

$$f_n(x) = \frac{x_{n+1} - x}{x_{n+1} - x_n} \qquad x_n \leq x \leq x_{n+1} \tag{3.36}$$

The impedance matrix elements can be written as

$$z_{mn} = \frac{k^2}{4\pi} \int_{z_n - \Delta_z/2}^{z_n + \Delta_z/2} \frac{e^{-jkR}}{R} \, dz' + \left[\frac{\partial}{\partial z'} \frac{e^{-jkR}}{R} \right] \Bigg|_{z' = z_n - \Delta_z/2}^{z' = z_n + \Delta_z/2} \tag{4.61}$$

where $R = \sqrt{(z_m - z')^2 + a^2}$, and $\partial/\partial z$ has been replaced by $\partial/\partial z'$. Evaluating the derivative in the second term yields

$$\frac{\partial}{\partial z'} \frac{e^{-jkR}}{R} = (z_m - z') \frac{1 + jkR}{R^3} e^{-jkR} \tag{4.62}$$

allowing us to write

$$z_{mn} = \frac{k^2}{4\pi} \int_{z_n - \Delta_z/2}^{z_n + \Delta_z/2} \frac{e^{-jkR}}{R} \, dz' + \left[(z_m - z') \frac{1 + jkR}{R^3} e^{-jkR} \right] \Bigg|_{z' = z_n - \Delta_z/2}^{z' = z_n + \Delta_z/2} \tag{4.63}$$

The matrix elements of (4.36) become

$$z_{mn} = \int_{z_n - \Delta_z/2}^{z_n + \Delta_z/2} \frac{e^{-jkR}}{4\pi R} \, dz' \tag{4.43}$$

where matching is done at the center of each segment z_m, and $R = \sqrt{(z_m - z')^2 + a^2}$. We will compute the non-self terms via an M-point numerical quadrature yielding

$$z_{mn} = \sum_{q=1}^{M} w_q \frac{e^{-jkR_{mq}}}{4\pi R_{mq}} \tag{4.44}$$

where $R_{mq} = \sqrt{(z_m - z_q)^2 + a^2}$. For the self terms ($m = n$), we will use a small-argument approximation to the Green's function to write

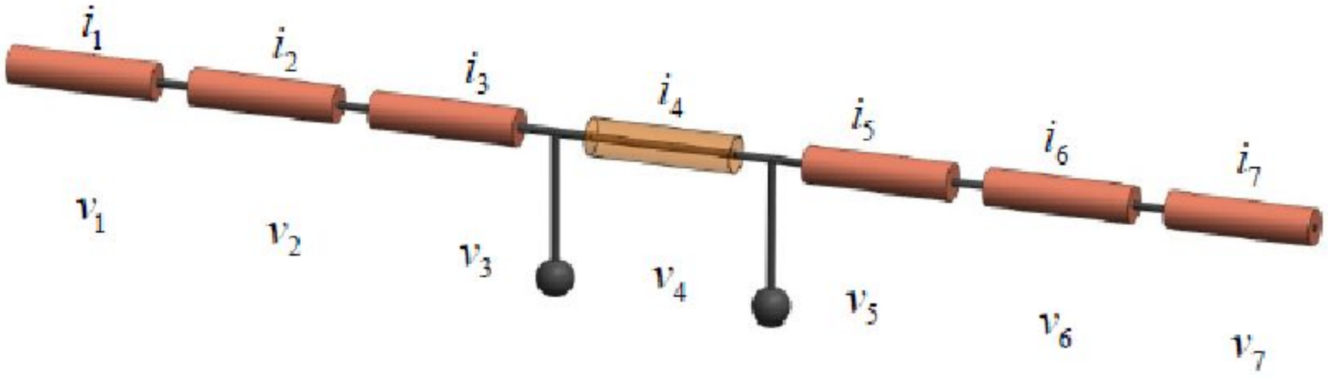
$$z_{mm} = \int_{-\Delta_z/2}^{\Delta_z/2} \frac{e^{-jkR}}{4\pi R} \, dz' \approx \int_{-\Delta_z/2}^{\Delta_z/2} \frac{1 - jkR}{4\pi R} \, dz' \tag{4.45}$$

which evaluates to [9] (Equation 200.01)

$$z_{mm} = \frac{1}{4\pi} \log \left[\frac{\sqrt{1 + 4a^2/\Delta_z^2} + 1}{\sqrt{1 + 4a^2/\Delta_z^2} - 1} \right] - \frac{jk\Delta_z}{4\pi} \tag{4.46}$$

Pulse Basis Functions (3 of 3)

We can now interpret [a] as a column vector containing the currents in each segment of the antenna.



$$[a]=[i]$$

$$[z_{mn}][a_n]=[g_m]$$

Transformation to True Impedance Matrix

The matrix equation is

$$[z_{mn}][a_n] = [g_m]$$

The a_n coefficients are the currents in each segment. The g_m coefficients are scaled electric fields. Based on this, it is more intuitive to write the matrix equation as

$$[z_{mn}][i_n] = \left[-j\omega\varepsilon E_z^{\text{inc}}(z_m)\right]$$

We would like the units on the right-hand side to be voltage so that the $[Z]$ matrix is true impedance. Voltage is related to the electric field through

$$E_z^{\text{inc}}(z_m) = \frac{V_m}{\Delta z}$$

The final matrix equation in terms of element voltage and current is

$$\frac{j\Delta z}{\omega\varepsilon}[z_{mn}][i_n] = [V_m] \qquad \underbrace{\frac{j\Delta z\eta}{k}}_{\text{True Z}}[z_{mn}][i_n] = [V_m]$$

Implementation

Initialize MATLAB

Define Simulation Parameters
- Wavelength, λ_0
- Antenna length, L
- Wire radius, a
- Atmosphere, μ_r and ε_r
- Number of segments, N

Compute Constants

$$k_0 = 2\pi/\lambda_0$$
$$k = k_0\sqrt{\mu_r \varepsilon_r}$$
$$\Delta z = L/N$$
$$\text{za} = [0:N-1]\cdot\Delta z$$

Build Impedance Matrix — See next slide

Transform to True Impedance

$$\mathbf{Z} = j\frac{\Delta z \eta}{k}\mathbf{Z}$$

Compute Admittance Matrix — Optional

$$\mathbf{Y} = \mathbf{Z}^{-1}$$

Compute Source Voltage

$$[v_m]^T = [0 \quad 0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad 0]$$

Compute Current

$$[i] = [z_{mn}]^{-1}[v]$$

Compute Input Impedance

$$Z_{\text{in}} = v_n / i_n \qquad \text{fed at segment } n$$

Compute Gain/Pattern

Building the Impedance Matrix

CEM

Compute Diagonal Term

$$z'_{mm} = \frac{1}{4\pi}\ln\left(\frac{\sqrt{1+\left(2a/\Delta z\right)^2}+1}{\sqrt{1+\left(2a/\Delta z\right)^2}-1}\right) - \frac{jk\Delta z}{4\pi}$$

Loop Over all m and n

Calculation Step #1

$$z'_{mn} = \begin{cases} z'_{mm} & \text{for } m = n \\ \displaystyle\int_{z_n-\frac{\Delta z}{2}}^{z_n+\frac{\Delta z}{2}} \frac{e^{-jkR}}{4\pi R}\,dz' & \text{for } m \neq n \end{cases}$$

Calculation Step #2

$$r_1 = \sqrt{\left(z_m - z_n + \Delta z/2\right)^2 + a^2}$$

$$t_1 = \left(z_m - z_n + \Delta z/2\right)\frac{1+jkr_1}{r_1^3}e^{-jkr_1}$$

$$r_2 = \sqrt{\left(z_m - z_n - \Delta z/2\right)^2 + a^2}$$

$$t_1 = \left(z_m - z_n - \Delta z/2\right)\frac{1+jkr_2}{r_2^3}e^{-jkr_2}$$

$$z_{mn} = k^2 z'_{mn} + t_2 - t_1$$