

# CEE 451G ENVIRONMENTAL FLUID MECHANICS

## LECTURE 1: SCALARS, VECTORS AND TENSORS

A **scalar** has magnitude but no direction.

An example is pressure  $p$ .

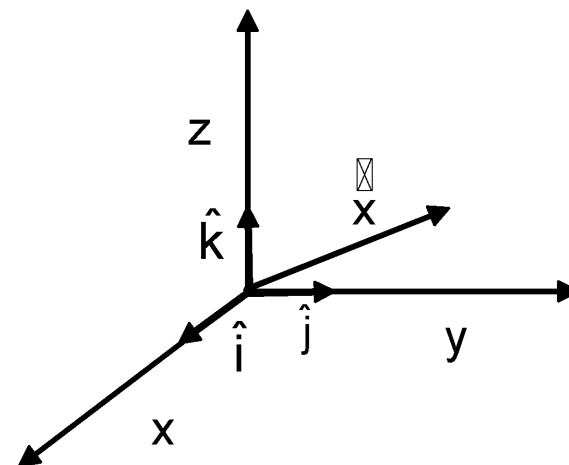
The coordinates  $x$ ,  $y$  and  $z$  of Cartesian space are scalars.

A **vector** has both magnitude and direction

Let  $\hat{i}$ ,  $\hat{j}$ ,  $\hat{k}$  denote **unit** vectors in the  $x$ ,  $y$  and  $z$  direction. The **hat** denotes a magnitude of unity

The **position vector**  $\vec{x}$  (the arrow denotes a vector that is not a unit vector) is given as

$$\vec{x} = x\hat{i} + y\hat{j} + z\hat{k}$$



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The **velocity vector**  $\vec{u}$  is given as

$$\vec{u} = \frac{d\vec{x}}{dt} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k}$$

The **acceleration vector**  $\vec{a}$  is given as

$$\vec{a} = \frac{d\vec{u}}{dt} = \frac{du}{dt} \hat{i} + \frac{dv}{dt} \hat{j} + \frac{dw}{dt} \hat{k} = \frac{d^2\vec{x}}{dt^2} = \frac{d^2x}{dt^2} \hat{i} + \frac{d^2y}{dt^2} \hat{j} + \frac{d^2z}{dt^2} \hat{k}$$

The **units** that we will use in class are length L, time T, mass M and temperature °. The units of a parameter are denoted in brackets. Thus

$$\begin{aligned} [\vec{x}] &= L \\ [\vec{u}] &= LT^{-1} \\ [\vec{a}] &= ? \quad LT^{-2} \end{aligned}$$

**Newton's second law** is a vectorial statement: where  $\vec{F}$  denotes the force vector and m denotes the mass (which is a scalar)

$$\vec{F} = m\vec{a}$$

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The components of the force vector can be written as follows:

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

The **dimensions** of the force vector are the dimension of mass times the dimension acceleration

$$[\vec{F}] = [F_x] = MLT^{-2}$$

Pressure  $p$ , which is a scalar, has dimensions of force per unit area. The dimensions of pressure are thus

$$[p] = MLT^{-2} / (L^2) = ML^{-1}T^{-2}$$

The acceleration of gravity  $g$  is a scalar with the dimensions of (of course) acceleration:

$$[g] = LT^{-2}$$

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A scalar can be a function of a vector, a vector of a scalar, etc. For example, in fluid flows pressure and velocity are both functions of position and time:

$$p = p(\vec{x}, t) \quad , \quad \vec{u} = \vec{u}(\vec{x}, t)$$

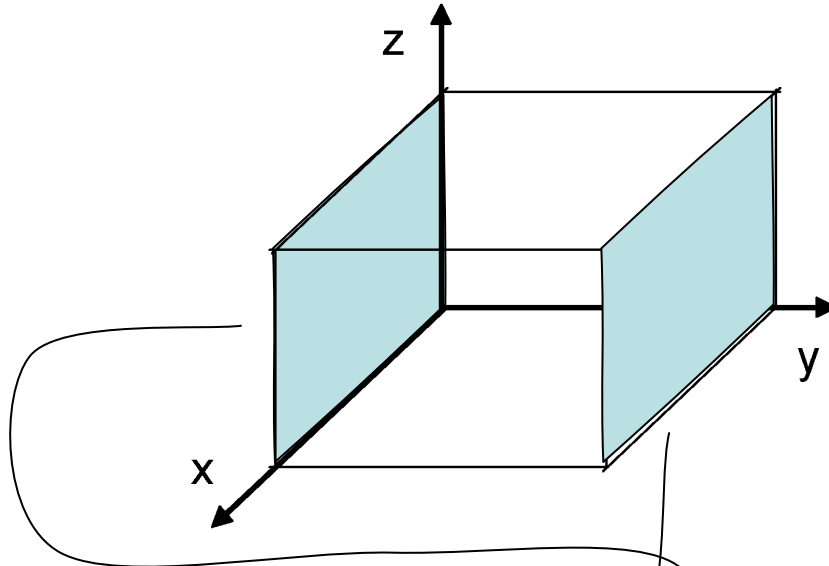
A scalar is a **zero-order tensor**. A vector is a **first-order tensor**. A matrix is a **second order tensor**. For example, consider the **stress tensor  $\tau$** .

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

The stress tensor has 9 components. What do they mean? Use the following mnemonic device: **first face, second stress**

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Consider the volume element below.



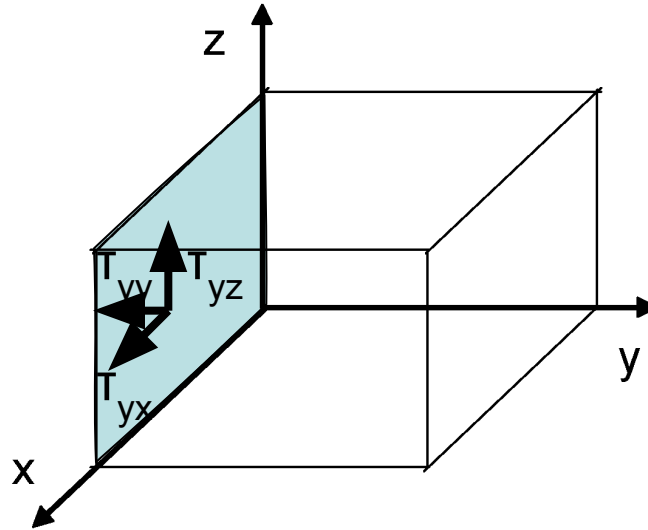
Each of the six faces has a **direction**.

For example, this face  
and this face  
are normal to the y direction

A force acting on any face can act in the x, y and z directions.

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Consider the face below.



The face is in the direction  $y$ .

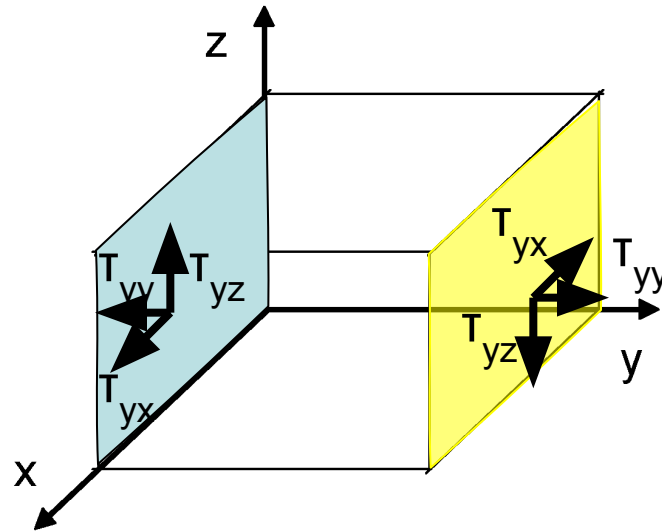
The force per unit face area acting in the  $x$  direction on that face is the stress  $\tau_{yx}$  (first face, second stress).

The forces per unit face area acting in the  $y$  and  $z$  directions on that face are the stresses  $\tau_{yy}$  and  $\tau_{yz}$ .

Here  $\tau_{yy}$  is a **normal stress** (acts normal, or perpendicular to the face) and  $\tau_{yx}$  and  $\tau_{yz}$  are **shear stresses** (act parallel to the face)

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Some conventions are in order



Normal stresses are defined to be positive **outward**, so the orientation is reversed on the face located  $\Delta y$  from the origin

Shear stresses similarly reverse sign on the opposite face face are the stresses  $\tau_{yy}$  and  $\tau_{yz}$ .

Thus a positive normal stress puts a body in tension, and a negative normal stress puts the body in compression. Shear stresses always put the body in shear.`

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Another way to write a vector is in **Cartesian** form:

$$\vec{x} = x\hat{i} + y\hat{j} + z\hat{k} = (x, y, z)$$

The coordinates  $x$ ,  $y$  and  $z$  can also be written as  $x_1$ ,  $x_2$ ,  $x_3$ . Thus the vector can be written as

$$\vec{x} = (x_1, x_2, x_3)$$

or as

$$\vec{x} = (x_i), i = 1..3$$

or in **index notation**, simply as

$$\vec{x} = x_i$$

where  $i$  is understood to be a dummy variable running from 1 to 3.

Thus  $x_i$ ,  $x_j$  and  $x_p$  all refer to the same vector  $(x_1, x_2 \text{ and } x_3)$ , as the index (subscript) always runs from 1 to 3.



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**Scalar multiplication:** let  $\alpha$  be a scalar and  $\vec{A} = A_i$  be a vector.

Then

$$\alpha\vec{A} = \alpha A_i = (\alpha A_1, \alpha A_2, \alpha A_3)$$

is a vector.

**Dot or scalar product** of two vectors results in a scalar:

$$\vec{A} \bullet \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = \text{scalar}$$

In index notation, the dot product takes the form

$$\vec{A} \bullet \vec{B} = \sum_{i=1}^3 A_i B_i = \sum_{k=1}^3 A_k B_k = \sum_{r=1}^3 A_r B_r =$$

Einstein summation convention: if the same index occurs twice, **always sum over that index**. So we abbreviate to

$$\vec{A} \bullet \vec{B} = A_i B_i = A_k B_k = A_r B_r$$

There is no free index in the above expressions. Instead the indices are paired (e.g. two  $i$ 's), implying summation. The result of the dot product is thus a scalar.

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**Magnitude** of a vector:

$$|\vec{A}|^2 = \vec{A} \cdot \vec{A} = A_i A_i$$

A **tensor** can be constructed by multiplying two vectors (not scalar product):

$$A_i B_j = (A_i B_j), i = 1..3, j = 1..3 = \begin{pmatrix} A_1 B_1 & A_2 B_1 & A_3 B_1 \\ A_1 B_2 & A_2 B_2 & A_3 B_2 \\ A_1 B_3 & A_2 B_3 & A_3 B_3 \end{pmatrix}$$

Two free indices (i, j) means the result is a **second-order** tensor

Now consider the expression

$$A_i A_j B_j$$

This is a **first-order tensor**, or **vector** because there is only one free index, i (the j's are paired, implying summation).

$$A_i A_j B_j = (A_1 B_1 + A_2 B_2 + A_3 B_2)(A_1, A_2, A_3)$$

That is, scalar times vector = vector.

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Kronecker delta  $\delta_{ij}$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since there are two free indices, the result is a second-order tensor, or matrix. The Kronecker delta corresponds to the **identity matrix**.

Third-order **Levi-Civita** tensor.

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ cycle clockwise: } 1, 2, 3, 2, 3, 1 \text{ or } 3, 1, 2 \\ -1 & \text{if } i, j, k \text{ cycle counterclockwise: } 1, 3, 2, 3, 2, 2 \text{ or } 2, 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

Vectorial **cross product**:

$$\vec{A} \times \vec{B} = \varepsilon_{ijk} A_j B_k$$

One free index, so the result must be a **vector**.

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Vectorial cross product: Let  $\vec{C}$  be given as

$$\vec{C} = \vec{A} \times \vec{B}$$

Then

$$\vec{C} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} =$$

$$\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \begin{pmatrix} \hat{i} & \hat{j} \\ A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} - \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \begin{pmatrix} \hat{i} & \hat{j} \\ A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} =$$

$$(A_2 B_3 - A_3 B_2) \hat{i} + (A_3 B_1 - A_1 B_3) \hat{j} + (A_1 B_2 - A_2 B_1) \hat{k}$$

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Vectorial cross product in tensor notation:

$$C_i = \varepsilon_{ijk} A_j B_k$$

Thus for example

$$C_1 = \varepsilon_{1jk} A_j B_k = \overset{= 1}{\varepsilon_{123}} A_2 B_3 + \overset{= -1}{\varepsilon_{132}} A_3 B_2 + \overset{= 0}{\varepsilon_{111}} A_1 B_1 + \text{a lot of other terms that all = 0}$$

$$= A_2 B_3 - A_3 B_2$$

i.e. the same result as the other slide. The same results are also obtained for  $C_2$  and  $C_3$ .

The **nabla vector operator**  $\nabla$  :

$$\nabla = \hat{i} \frac{\partial}{\partial x_1} + \hat{j} \frac{\partial}{\partial x_2} + \hat{k} \frac{\partial}{\partial x_3}$$

or in index notation

$$\frac{\partial}{\partial x_i}$$

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The **gradient** converts a scalar to a vector. For example, where  $p$  is pressure,

$$\text{grad}(p) = \nabla p = \frac{\partial p}{\partial x_1} \hat{i} + \frac{\partial p}{\partial x_2} \hat{j} + \frac{\partial p}{\partial x_3} \hat{k}$$

or in index notation

$$\text{grad}(p) = \frac{\partial p}{\partial x_i}$$

The single free index  $i$  (free in that it is not paired with another  $i$ ) in the above expression means that  $\text{grad}(p)$  is a vector.

The **divergence** converts a vector into a scalar. For example, where  $\mathbf{u}$  is the velocity vector,

$$\text{div}(\mathbf{u}) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_k}{\partial x_k}$$

Note that there is no free index (two  $i$ 's or two  $k$ 's), so the result is a scalar.

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The **curl** converts a vector to a vector. For example, where  $\vec{u}$  is the velocity vector,

$$\text{curl}(\vec{u}) = \nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} =$$
$$\left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \hat{i} + \left( \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \hat{j} + \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \hat{k}$$

or in index notation,

$$\text{curl}(\vec{u}) = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

One free index  $i$  (the  $j$ 's and the  $k$ 's are paired) means that the result is a vector

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A useful manipulation in tensor notation can be used to change an index in an expression:

$$\delta_{ij}u_j = u_i$$

This manipulation works because the Kronecker delta  $\delta_{ij} = 0$  except when  $i = j$ , in which case it equals 1.