Calculus++ Light

Question 1. A sequence a_n, $n = 1,2,3,\ldots,$ satisfies $\lim_{n\to\infty}(2n-1)a_n = 16.$

a) Use the definition of limit to obtain a sandwich inequality for a_n.

Solution: Since the limit of $(2n-1)\, a_n$ is 16 we have: $\forall \varepsilon > 0, \exists N, \forall n > N : |(2n-1)a_n - 16| < \varepsilon.$

Set $\varepsilon = 1$, then $|(2n-1)a_n - 16| < 1, \ \forall n > N.$

That is, $-1 < (2n-1)a_n - 16 < 1$

$$\Leftrightarrow \quad 15 < (2n-1)a_n < 17$$

$$\Leftrightarrow \quad \frac{15}{2n-1} < a_n < \frac{17}{2n-1} \quad \Rightarrow \quad \lim_{n\to\infty} a_n = 0.$$

b) Conclude that $\lim_{n\to\infty} a_n = 0$,

and find $\lim_{n\to\infty} n a_n$.

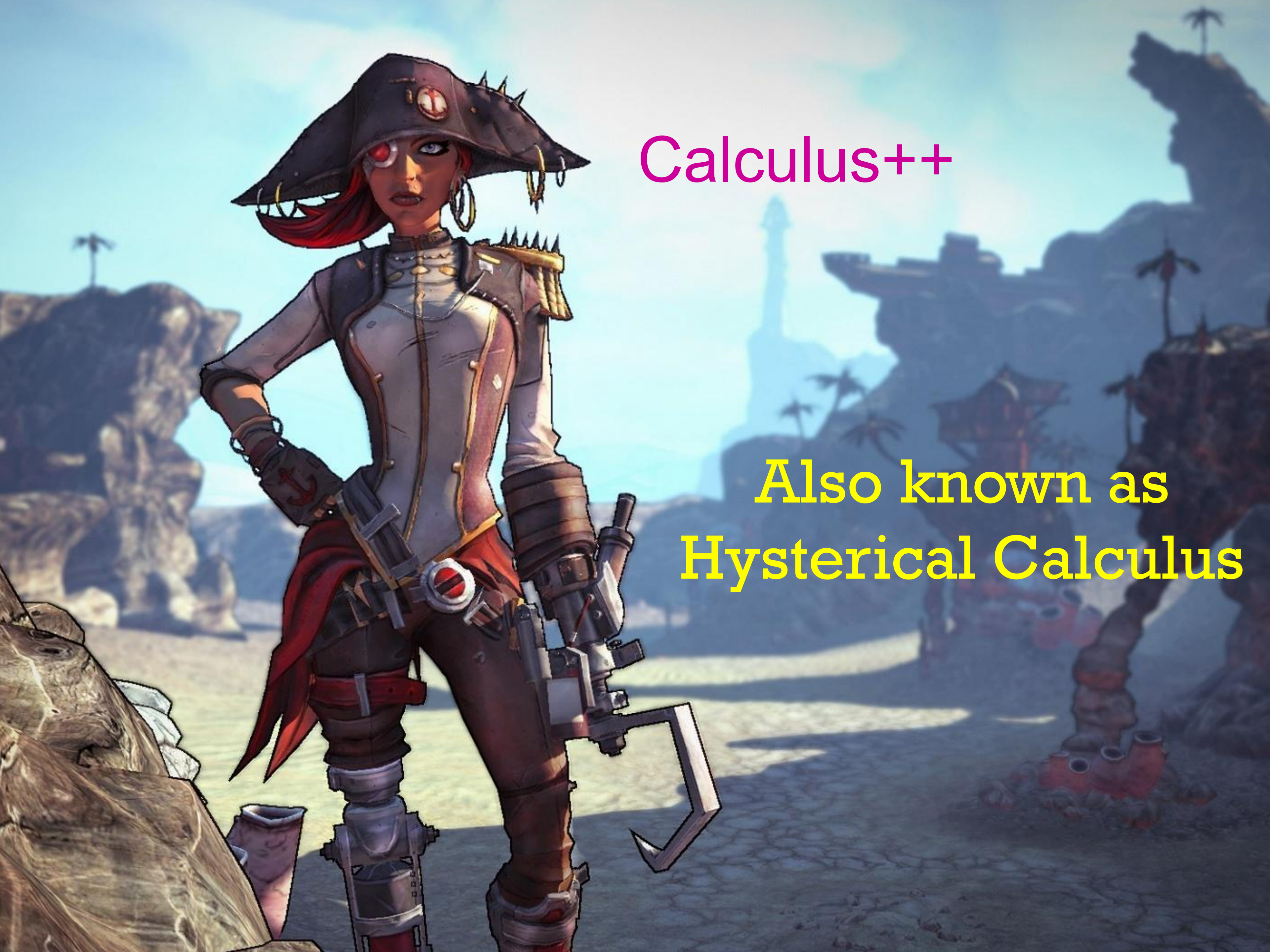
We have $n a_n = \frac{1}{2}(2n-1)a_n + \frac{1}{2}a_n$

Therefore

$$\lim_{n\to\infty} n a_n = \lim_{n\to\infty} \tfrac{1}{2}(2n-1)a_n + \lim_{n\to\infty} \tfrac{1}{2}a_n$$

$$= \tfrac{1}{2}\lim_{n\to\infty}(2n-1)a_n + \tfrac{1}{2}\lim_{n\to\infty} a_n$$

$$= \tfrac{1}{2}\cdot 16 + \tfrac{1}{2}\cdot 0 = 8.$$



Calculus++

Also known as
Hysterical Calculus

Question 2. A sequence x_n, $n = 1,2,3,\ldots$ is defined by the relationship $x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$ and the initial conditions $x_1 = a$, $x_2 = b$.

Find $\lim\limits_{n\to\infty} x_n$.

Solution. We begin with finding an explicit expression for the general term of the sequence x_n.

Let us try the following formula: $x_n = c\lambda^n$.

$$x_n = \frac{x_{n-1} + x_{n-2}}{2} \Rightarrow c\lambda^n = \tfrac{1}{2}c\lambda^{n-1} + \tfrac{1}{2}c\lambda^{n-2}.$$

Divide both sides by $c\lambda^{n-2}$ to obtain $\lambda^2 = \tfrac{1}{2}\lambda + \tfrac{1}{2}$,

or $\lambda^2 - \tfrac{1}{2}\lambda - \tfrac{1}{2} = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = -\tfrac{1}{2}.$

Thus, we found two sequences that satisfy the defining relationship $x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$:

$x_n^{(1)} = c_1$ and $x_n^{(2)} = c_2(-\frac{1}{2})^n$.

Do any of these sequences satisfy the initial conditions $x_1 = a$, $x_2 = b$?

Well, if $a = b$, then the first sequence with $c_1 = a$, $x_n^{(1)} = a$, satisfies the initial conditions.

If $b = -½\, a$, then the second sequence with $c_2 = -2\,a$, $x_n^{(2)} = a(-\frac{1}{2})^{n-1}$, satisfies the initial conditions.

But what should we do if a and b are arbitrary?

Well, we can consider linear combination of the two obtained sequences $x_n = c_1 + c_2(-\frac{1}{2})^n$.
Let us check that this linear combination indeed satisfies the equation $x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$.
We have

$$\frac{x_{n-1} + x_{n-2}}{2} = \frac{c_1 + c_2(-\frac{1}{2})^{n-1} + c_1 + c_2(-\frac{1}{2})^{n-2}}{2}$$

$$= c_1 + c_2(-\tfrac{1}{2})^n \frac{(-\frac{1}{2})^{-1} + (-\frac{1}{2})^{-2}}{2}$$

$$= c_1 + c_2(-\tfrac{1}{2})^n = x_n.$$

Now all we have to do is to find the values of c_1 and c_2 such that our sequence also satisfies the initial conditions:

$$x_1 = c_1 + c_2(-\tfrac{1}{2}) = a,$$
$$x_2 = c_1 + \tfrac{1}{4}c_2 = b.$$

For the values of arbitrary constants c_1 and c_2 we obtain $3c_1 = a + 2b,\ \tfrac{3}{4}c_2 = b - a.$

Thus $$x_n = \frac{a+2b}{3} + \frac{4}{3}(b-a)(-\tfrac{1}{2})^n.$$

Now the limit $n \to \infty$ is not difficult to find:

$$\lim_{n\to\infty} x_n = \tfrac{1}{3}a + \tfrac{2}{3}b.$$

The method of the week

To find the sequence that satisfies the defining relationship $x_n = \beta x_{n-1} + \gamma x_{n-2}$,
and the initial conditions $x_1 = a, x_2 = b$ we have to:

1. Write down the characteristic equation

$$\lambda^2 - \beta \lambda - \gamma = 0$$

and obtain its roots λ_1, λ_2.

2. Write down the general formula for x_n:

$$x_n = c_1 \lambda_1^n + c_2 \lambda_2^n,$$

and find the values of constants c_1 and c_2, such that $x_1 = a, x_2 = b$.

Question 3 a). Find the following limit

$$\lim_{n\to\infty} \sin\left(\pi\sqrt{n^2+1}\right).$$

Solution: We have

$$\sin\left(\pi\sqrt{n^2+1}\right) = \sin\left(\pi\sqrt{n^2+1} - \pi n + \pi n\right)$$

$$= \sin\left(\pi\sqrt{n^2+1} - \pi n\right)\cos(\pi n) +$$

$$+ \cos\left(\pi\sqrt{n^2+1} - \pi n\right)\underbrace{\sin(\pi n)}_{=0}$$

$$= \sin\left(\pi\sqrt{n^2+1} - \pi n\right)(-1)^n.$$

We have $\sqrt{n^2+1}-n=$

$$=\left(\sqrt{n^2+1}-n\right)\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}=\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

$$=\frac{1}{\sqrt{n^2+1}+n}.$$

The obtained identity yields

$$\sin\left(\pi\sqrt{n^2+1}\right)=\sin\left(\pi\sqrt{n^2+1}-\pi n\right)(-1)^n$$

$$=\sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right)(-1)^n.$$

Therefore we can use the following sandwich inequality

$$-\sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right) \le \sin\left(\pi\sqrt{n^2+1}\right) \le$$
$$\le \sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right).$$

Since $\sin x$ is a continuous function we obtain

$$\lim_{n\to\infty}\sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right) = \sin\left(\lim_{n\to\infty}\frac{\pi}{\sqrt{n^2+1}+n}\right)$$
$$= \sin 0 = 0.$$

Hence, the sandwich theorem tells us that

$$\lim_{n\to\infty}\sin\left(\pi\sqrt{n^2+1}\right) = 0.$$

Question 4. State a (positive) definition of a divergent sequence $\{x_n\}$.

Solution: We begin with the definition of a convergent sequence.

A sequence $\{x_n\}$ converges to a number L, if

$$\forall \varepsilon > 0, \exists N, \forall n > N : | x_n - L | < \varepsilon.$$

A sequence $\{x_n\}$ does not converges to a number L, if

$$\exists \varepsilon > 0, \forall N, \exists n > N : | x_n - L | > \varepsilon.$$

A sequence $\{x_n\}$ is divergent, if it does not converges to any number L.

$$\forall L, \exists \varepsilon > 0, \forall N, \exists n > N : | x_n - L | > \varepsilon.$$

Question 5. Draw the curve defined by the equation $\lim_{n\to\infty}\sqrt[n]{|x|^n+|y|^n}=1$ in the $xy-$plane.

Solution. To begin with, we calculate the limit

$$\lim_{n\to\infty}\sqrt[n]{|x|^n+|y|^n},$$

in the particular case $x=-7, y=5.$

We have $|-7|^n \le |-7|^n+|5|^n \le 2|-7|^n.$

Hence, $|-7| \le \sqrt[n]{|-7|^n+|5|^n} \le \sqrt[n]{2}\,|-7|.$

Since $\lim_{n\to\infty}\sqrt[n]{2}=1,$ the sandwich theorem tells us that $\lim_{n\to\infty}\sqrt[n]{|-7|^n+|5|^n}=|-7|=7.$

Now we can find the limit $\lim_{n\to\infty} \sqrt[n]{|x|^n + |y|^n}$.

Note the following double inequality

$$\left(\max(|x|,|y|)\right)^n \le |x|^n + |y|^n \le$$

$$\le 2\left(\max(|x|,|y|)\right)^n.$$

Hence

$$\max(|x|,|y|) \le \sqrt[n]{|x|^n + |y|^n} \le$$

$$\le \sqrt[n]{2}\max(|x|,|y|).$$

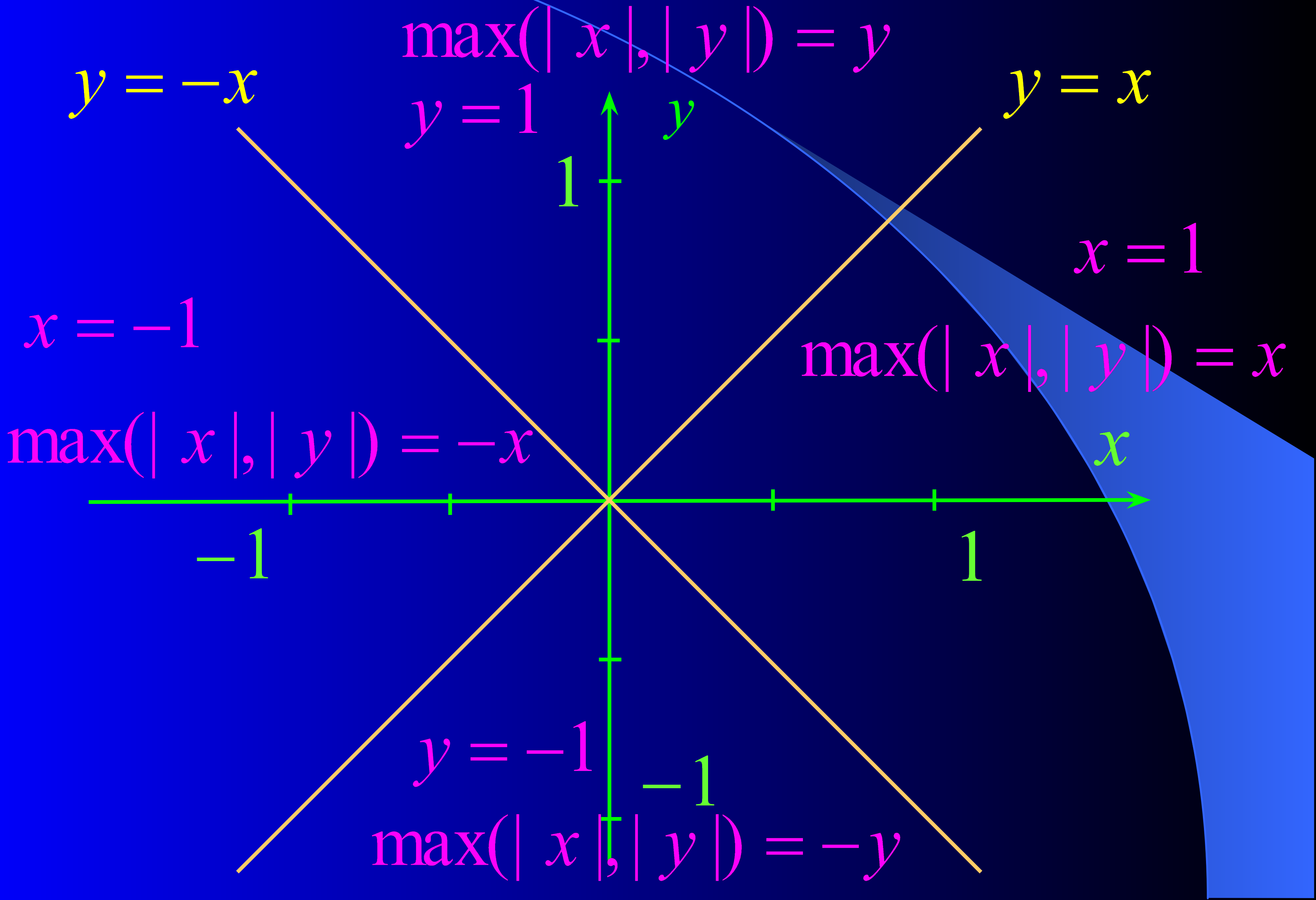
Since $\lim_{n\to\infty}\sqrt[n]{2}=1,$ the sandwich theorem tells us that

$$\lim_{n\to\infty}\sqrt[n]{|x|^n+|y|^n}=\max(|x|,|y|).$$

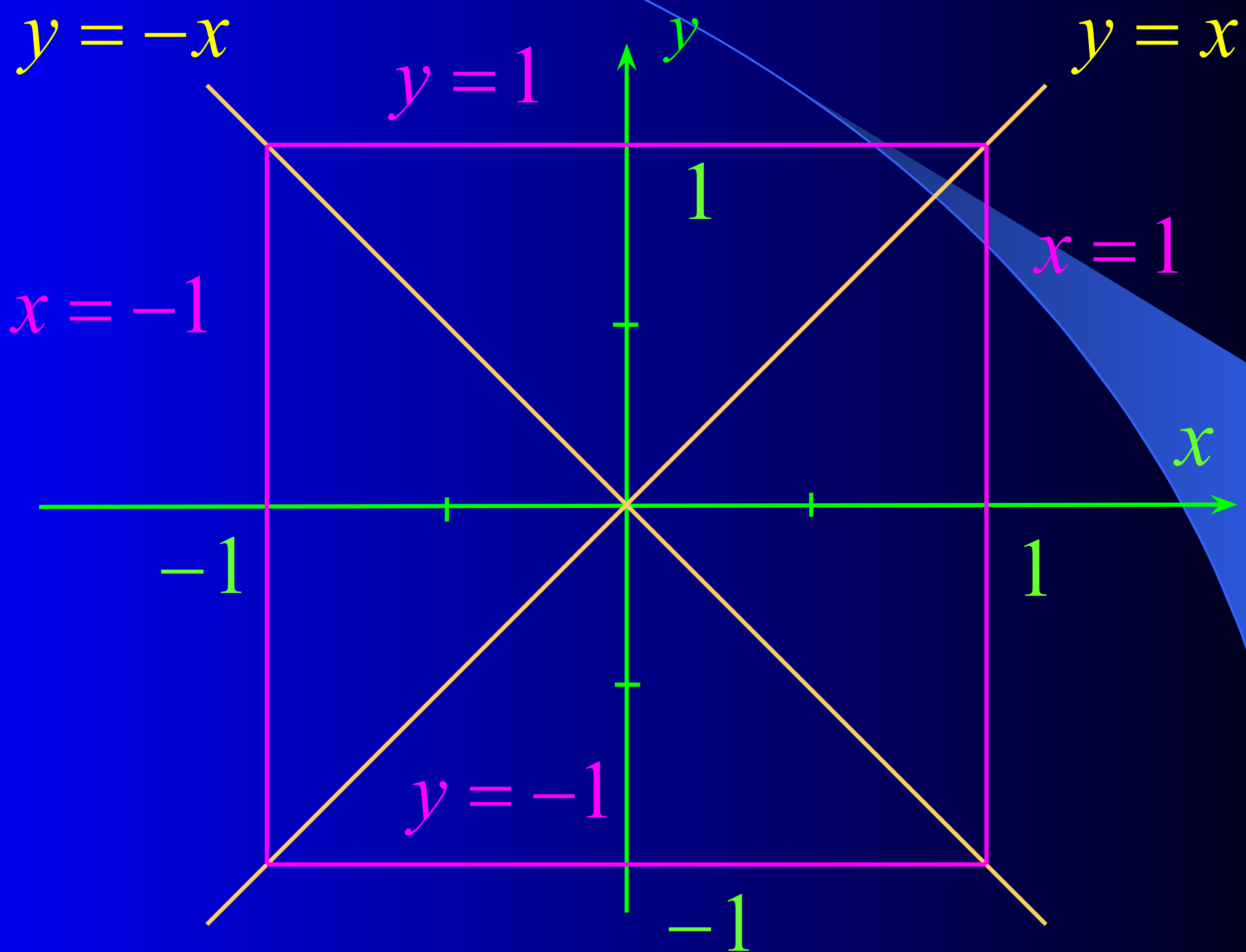
Thus, we have to draw the curve defined by the equation

$$\max(|x|,|y|)=1.$$

Let us look at the $xy-$plane:



The graph of the curve $\lim_{n\to\infty} \sqrt[n]{|x|^n + |y|^n} = 1.$



Question 6. Use the definition of convergent sequence to obtain a sandwich inequality for the sequence

$$x_n = \left(\frac{1007}{1008} + \frac{\sin n}{n}\right)^n, \quad n = 1,2,3,...,$$

and find the limit of this sequence.

Solution: The sequence $y_n = \frac{\sin n}{n}$ converges to 0.

Therefore, according to the definition of the limit, $\forall \varepsilon > 0, \exists N, \forall n > N : |y_n - 0| < \varepsilon.$

Choose $\varepsilon = \frac{1}{2016}$ and denote N_1 – the corresponding value of N.

The definition tells us that

$$-\frac{1}{2016} \le \frac{\sin n}{n} \le \frac{1}{2016} \quad \text{for all} \quad n > N_1.$$

$$\Rightarrow \frac{1007}{1008} - \frac{1}{2016} \le \frac{1007}{1008} + \frac{\sin n}{n} \le \frac{1007}{1008} + \frac{1}{2016}$$

$$\Rightarrow \frac{2013}{2016} \le \frac{1007}{1008} + \frac{\sin n}{n} \le \frac{2015}{2016}$$

Therefore we obtain the following sandwich inequality for our sequence x_n

$$\left(\frac{2013}{2016}\right)^n \le \left(\frac{1007}{1008} + \frac{\sin n}{n}\right)^n \le \left(\frac{2015}{2016}\right)^n$$

0   0

for all $n > N_1$.

Now the sandwich theorem tells us that

$$\lim_{n\to\infty}\left(\frac{1007}{1008}+\frac{\sin n}{n}\right)^n = 0.$$