

Learning Objectives for Section 4.5

Inverse of a Square Matrix



- The student will be able to identify identity matrices for multiplication.
- The student will be able to find the inverse of a square matrix.
- The student will be able to work with applications of inverse matrices such as cryptography.

Identity Matrix for Multiplication

- 1 is called the **multiplicative identity** for real numbers since

$$a(1) = (1)a = a$$

For example $5(1) = 5$

A matrix is called **square** if it has the same number of rows and columns, that is, it has size $n \times n$.

The set of all square matrices of size $n \times n$ also has a multiplicative identity I_n , with the property

$$AI_n = I_n A = A$$

I_n is called the $n \times n$ **identity matrix**.

Identity Matrices

■ 2 x 2 identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

■ 3 x 3 identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity Matrix Multiplication

- $AI = A$ (Verify the multiplication)
- We can also show that $IA = A$ and in general $AI = IA = A$ for all square matrices A .

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Inverse of a Matrix



- All real numbers (excluding 0) have an inverse.

$$a \cdot \frac{1}{a} = 1$$

For example

$$5 \cdot \frac{1}{5} = 1$$

Matrix Inverses

- Some (not all) square matrices also have matrix inverses
- If the inverse of a matrix A exists, we shall call it A^{-1}
- Then

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n$$

Inverse of a 2x2 Matrix

- There is a simple procedure to find the inverse of a two by two matrix. This procedure **only works for the 2 x 2 case**.
- An example will be used to illustrate the procedure.
- **Example:** Find the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

Inverse of a 2x2 matrix (continued)

- **Step 1:** Determine whether or not the inverse actually exists. We define Δ = the difference of the product of the diagonal elements of the matrix.

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

- In order for the inverse of a 2 x 2 matrix to exist, Δ cannot equal zero.
- If Δ happens to be zero, then we conclude the inverse does not exist, and we stop all calculations.
- In our case $\Delta = 2(2) - 1(3) = 1$, so we can proceed.

Inverse of a 2x2 matrix (continued)

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Step 2. Reverse the entries on the main diagonal.
In this example, both entries are 2, and no change is visible.

Step 3. Reverse the signs of the other diagonal entries 3 and 1 so they become -3 and -1

Step 4. Divide each element of the matrix by which in this case is 1, so no apparent change will be noticed.

Inverse of a 2x2 matrix (continued)

- The inverse of the matrix is then $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

- To verify that this is the inverse, we will multiply the original matrix by its inverse and hopefully obtain the 2 x 2 identity matrix:

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse of a General Square Matrix



1. Augment the matrix with the $n \times n$ identity matrix.
2. Use elementary row operations to transform the matrix on the left side of the vertical line to the $n \times n$ identity matrix. The row operations are used for the **entire row**, so that the matrix on the right hand side of the vertical line will also change.
3. When the matrix on the left is transformed to the $n \times n$ identity matrix, the matrix on the right of the vertical line is the inverse.

Example:

Inverse of a 3x3 Matrix

■ Find the inverse of
$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix}$$

- **Step 1.** Multiply R1 by (-2) and add the result to R2.
- **Step 2.** Multiply R1 by 2 and add the result to R3

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & -1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{r_2 - 2r_1 = R_2 \\ r_3 + 2r_1 = R_3}} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & \textcircled{3} & -4 & -2 & 1 & 0 \\ 0 & -4 & 7 & 2 & 0 & 1 \end{array} \right] \xrightarrow{\frac{1}{3}r_2 = R_2} \text{TO NEXT LINE}$$

Example (continued)

- **Step 3.** Multiply row 2 by $(1/3)$ to get a 1 in the second row, first position.
- **Step 4.** Add R_2 to R_1 .
- **Step 5.** Multiply R_2 by 4 and add the result to R_3 .
- **Step 6.** Multiply R_3 by $3/5$ to get a 1 in the third row, third position.

FROM ABOVE LINE

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -4 & 7 & 2 & 0 & 1 \end{array} \right] \xrightarrow{\substack{r_1 + r_2 = R_1 \\ r_3 + 4r_2 = R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{5}{3} & -\frac{2}{3} & \frac{4}{3} & 1 \end{array} \right] \xrightarrow{\frac{3}{5}r_3 = R_3} \text{TO NEXT LINE}$$

Example (continued)

- **Step 7.** Eliminate the $(5/3)$ in the first row, third position by multiplying R3 by $(-5/3)$ and adding result to R1.
- **Step 8.** Eliminate the $(-4/3)$ in the second row, third position by multiplying R3 by $(4/3)$ and adding result to R2.
- **Step 9.** You now have the identity matrix on the left, which is our goal.

FROM ABOVE LINE

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{4}{5} & \frac{3}{5} \end{array} \right] \xrightarrow{\begin{array}{l} r_1 - \frac{5}{3}r_3 = R_1 \\ r_2 + \frac{4}{3}r_3 = R_2 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -\frac{6}{5} & \frac{7}{5} & \frac{4}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{4}{5} & \frac{3}{5} \end{array} \right]$$

Example Solution

- The inverse matrix appears on the right hand side of the vertical line and is displayed below. Many calculators as well as computers have software programs that can calculate the inverse of a matrix quite easily. If you have access to a TI 83, consult the manual to determine how to find the inverse using a calculator.

$$\begin{bmatrix} 1 & -1 & -1 \\ -\frac{6}{5} & \frac{7}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

Application: Cryptography



Matrix inverses can provide a simple and effective procedure for encoding and decoding messages. To begin, assign the numbers 1-26 to the letters in the alphabet, as shown below. Also assign the number 0 to a blank to provide for space between words.

Blank	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

Thus the message “SECRET CODE” corresponds to the sequence

19 5 3 18 5 20 0 3 15 4 5

Cryptography (continued)

Any matrix A whose elements are positive integers and whose inverse exists can be used as an **encoding matrix**.

For example, to use the 2×2 matrix

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

to encode the message above, first divide the numbers in the sequence 19 5 3 18 5 20 0 3 15 4 5 into groups of 2, and use these groups as the columns of a matrix B :

$$B = \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{bmatrix}$$

We added an extra blank at the end of the message to make the columns come out even.

Cryptography (continued)

Then we multiply this matrix on the left by A :

$$\begin{aligned} AB &= \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 91 & 66 & 80 & 9 & 72 & 20 \\ 24 & 21 & 25 & 3 & 19 & 5 \end{bmatrix} \end{aligned}$$

The coded message is

$$91 \ 24 \ 66 \ 21 \ 80 \ 25 \ 9 \ 3 \ 72 \ 19 \ 20 \ 5$$

This message can be decoded simply by putting it back into matrix form and multiplying on the left by the decoding matrix A^{-1} .