Learning Objectives for Section 4.4 Matrices: Basic Operations

- The student will be able to perform addition and subtraction of matrices.
- The student will be able to find the scalar product of a number k and a matrix M.
- The student will be able to calculate a matrix product.

Addition and Subtraction of Matrices

- To add or subtract matrices, they must be of the same order, *m* x *n*. To add matrices of the same order, add their corresponding entries. To subtract matrices of the same order, subtract their corresponding entries. The general rule is as follows using mathematical notation:

$$A + B = \left[a_{ij} + b_{ij} \right]$$

$$A - B = \left[a_{ij} - b_{ij} \right]$$

Example: Addition

- Add the matrices

$$\begin{bmatrix} 4 & -3 & 1 \\ 0 & 5 & -2 \\ 5 & -6 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$$

Example: Addition Solution

- Add the matrices

$$\begin{bmatrix} 4 & -3 & 1 \\ 0 & 5 & -2 \\ 5 & -6 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$$

- **Solution:** First note that each matrix has dimensions of 3x3, so we are able to perform the addition. The result is shown at right:

- Adding corresponding entries, we have

$$\begin{bmatrix} 3 & -1 & 4 \\ 6 & -2 & 7 \\ 5 & -10 & 8 \end{bmatrix}$$

Example: Subtraction

- Now, we will subtract the same two matrices

$$\begin{bmatrix} 4 & -3 & 1 \\ 0 & 5 & -2 \\ 5 & -6 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$$

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Example: Subtraction Solution

- Now, we will subtract the same two matrices

$$\begin{bmatrix} 4 & -3 & 1 \\ 0 & 5 & -2 \\ 5 & -6 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$$

- Subtract corresponding entries as follows:

$$\begin{bmatrix} 4-(-1) & -3-2 & 1-3 \\ 0-6 & 5-(-7) & -2-9 \\ 5-0 & -6-(-4) & 0-8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 & -2 \\ -6 & 12 & -11 \\ 5 & -2 & -8 \end{bmatrix}$$

Scalar Multiplication

- The **scalar product** of a number *k* and a matrix A is the matrix denoted by *k*A**,** obtained by multiplying each entry of A by the number *k*. The number *k* is called a **scalar.** In mathematical notation,

$$kA = \left[ka_{ij} \right]$$

Example: Scalar Multiplication

- Find (-1)A, where A =

$$\begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$$

Example: Scalar Multiplication Solution

- Find (-1)A, where A =

$$\begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$$

- **Solution:**

$$(-1)A = -1 \begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix}$$

$$= (-1)\begin{bmatrix} -1 & 2 & 3 \\ 6 & -7 & 9 \\ 0 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -6 & 7 & -9 \\ 0 & 4 & -8 \end{bmatrix}$$

Alternate Definition of Subtraction of Matrices

- The definition of subtraction of two real numbers *a* and *b* is $a - b = a + (-1)b$ or "*a* plus the opposite of *b*". We can define subtraction of matrices similarly:
- If *A* and *B* are two matrices of the same dimensions, then

  $$A - B = A + (-1)B,$$

  where (-1) is a scalar.

Example

- The example on the right illustrates this procedure for two 2x2 matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + (-1)\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 0 & 5 \end{bmatrix}$$

Matrix Equations

Example: Find a, b, c, and d so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

Matrix Equations

Example: Find a, b, c, and d so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

Solution: Subtract the matrices on the left side:

$$\begin{bmatrix} a-2 & b+1 \\ c+5 & d-6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

Use the definition of equality to change this matrix equation into 4 real number equations:

a - 2 = 4 $\quad$ b + 1 = 3 $\quad$ c + 5 = -2 $\quad$ d - 6 = 4 $\quad$ a = 6

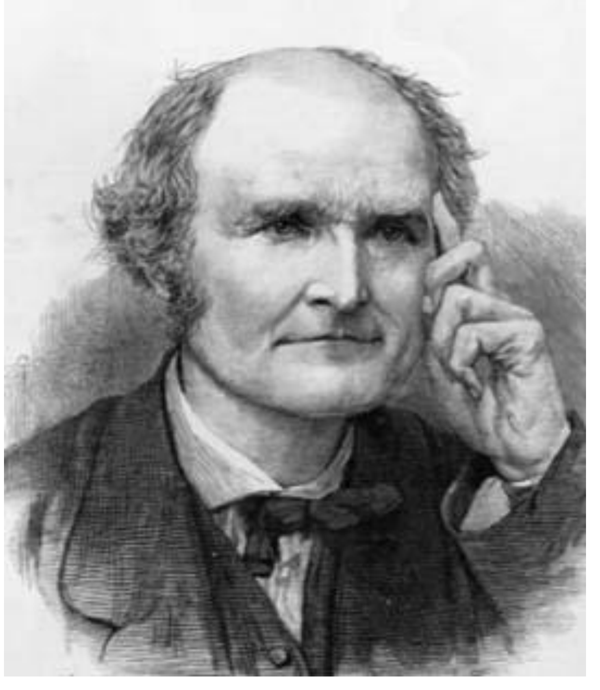
b = 2 $\quad$ c = -7 $\quad$ d = 10

Matrix Products

- The method of multiplication of matrices is not as intuitive and may seem strange, although this method is extremely useful in many mathematical applications.
- Matrix multiplication was introduced by an English mathematician named Arthur Cayley (1821-1895). We will see shortly how matrix multiplication can be used to solve systems of linear equations.

Arthur Cayley
1821-1895

- Introduced matrix multiplication



Product of a Row Matrix and a Column Matrix

- In order to understand the general procedure of matrix multiplication, we will introduce the concept of the product of a row matrix by a column matrix.
- A **row matrix** consists of a single row of numbers, while a **column matrix** consists of a single column of numbers. If the number of columns of a row matrix equals the number of rows of a column matrix, the product of a row matrix and column matrix is defined. Otherwise, the product is not defined.

Row by Column Multiplication

- **Example:** A row matrix consists of 1 row of 4 numbers so this matrix has four columns. It has dimensions 1 x 4. This matrix can be multiplied by a column matrix consisting of 4 numbers in a single column (this matrix has dimensions 4 x 1).
- 1x4 row matrix multiplied by a 4x1 column matrix. Notice the manner in which corresponding entries of each matrix are multiplied:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} = 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 = 70$$

Example: Revenue of a Car Dealer

- A car dealer sells four model types: A, B, C, D. In a given week, this dealer sold 10 cars of model A, 5 of model B, 8 of model C and 3 of model D. The selling prices of each automobile are respectively \$12,500, \$11,800, \$15,900 and \$25,300. Represent the data using matrices and use matrix multiplication to find the total revenue.

Solution using Matrix Multiplication

- We represent the number of each model sold using a row matrix (4x1), and we use a 1x4 column matrix to represent the sales price of each model. When a 4x1 matrix is multiplied by a 1x4 matrix, the result is a 1x1 matrix containing a single number.

$$\begin{bmatrix} 10 & 5 & 8 & 3 \end{bmatrix} \begin{bmatrix} 12,500 \\ 11,800 \\ 15,900 \\ 25,300 \end{bmatrix} = \begin{bmatrix} 10(12,500) + 5(11,800) + 8(15,900) + 3(25,300) \end{bmatrix} = \begin{bmatrix} 387,100 \end{bmatrix}$$

Matrix Product

- If A is an m x p matrix and B is a p x n matrix, the matrix product of A and B, denoted by AB, is an m x n matrix whose element in the i th row and j th column is the real number obtained from the product of the i th row of A and the j th column of B. If the number of columns of A does not equal the number of rows of B, the matrix product AB is not defined.

Multiplying a 2x4 matrix by a 4x3 matrix to obtain a 2x3

- The following is an illustration of the product of a 2x4 matrix with a 4x3. First, the number of columns of the matrix on the left must equal the number of rows of the matrix on the right, so matrix multiplication is defined. A row-by column multiplication is performed three times to obtain the first row of the product: 70 80 90.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 70 & 80 & 90 \\ 158 & 184 & 210 \end{pmatrix}$$

Final Result

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 70 & 80 & 90 \\ 158 & 184 & 210 \end{pmatrix}$$

Undefined Matrix Multiplication

Why is the matrix multiplication below not defined?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \text{ is not defined}$$

Undefined Matrix Multiplication Solution

Why is the matrix multiplication below not defined?

The answer is that the left matrix has three columns but the matrix on the right has only two rows. To multiply the second row [4 5 6] by the third column, $\begin{bmatrix}3\\7\end{bmatrix}$, there is no number to pair with 6 to multiply.

$$\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\\10 & 11 & 12\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{pmatrix}$$ is not defined

Example

Given $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}$

Find AB if it is defined:

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}$$

Solution

- Since *A* is a 2 x 3 matrix and *B* is a 3 x 2 matrix, *AB* will be a 2 x 2 matrix.

  1. Multiply first row of *A* by first column of *B*: 3(1) + 1(3) +(-1)(-2)=8

  2. First row of *A* times second column of B: 3(6)+1(-5)+ (-1)(4)= 9

  3. Proceeding as above the final result is

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 9 \\ -4 & 24 \end{bmatrix}$$

Is Matrix Multiplication Commutative?

- Now we will attempt to multiply the matrices in reverse order: $BA = ?$
- We are multiplying a 3 x 2 matrix by a 2 x 3 matrix. This matrix multiplication is defined, but the result will be a 3 x 3 matrix. Since AB does not equal BA, matrix multiplication is **not commutative.**

$$\begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 1 & 17 \\ -1 & 3 & -18 \\ 2 & -2 & 14 \end{bmatrix}$$

Practical Application

- Suppose you a business owner and sell clothing. The following represents the number of items sold and the cost for each item. Use matrix operations to determine the total revenue over the two days:

  Monday: 3 T-shirts at $10 each, 4 hats at $15 each, and 1 pair of shorts at $20.
  Tuesday: 4 T-shirts at $10 each, 2 hats at $15 each, and 3 pairs of shorts at $20.

Solution of Practical Application

- Represent the information using two matrices: The product of the two matrices gives the total revenue:

Unit price of each item:

Qty sold of each item on Monday

$$\begin{bmatrix} 10 & 15 & 20 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 2 \\ 1 & 3 \end{bmatrix}$$

Qty sold of each item on Tuesday

- Then your total revenue for the two days is $= \begin{bmatrix} 110 & 130 \end{bmatrix}$
Price times Quantity = Revenue