NUFYP Mathematics & Computing Science

Pre-Calculus course

1.5 Basic transformations of graphs

Lecture slides prepared by: Viktor Ten

Lecture overview: learning outcomes

At the end of this lecture, you should be able to:

1.5.1 Sketch the graph of a cubic function given in factorized form.

1.5.2 Apply a horizontal translation to a given curve.

1.5.3 Apply a vertical translation to a given curve.

1.5.4 Apply a vertical stretch to a given curve.

1.5.5 Apply a horizontal stretch to a given curve.

1.5.6 Apply simple combined transformations to a given curve.

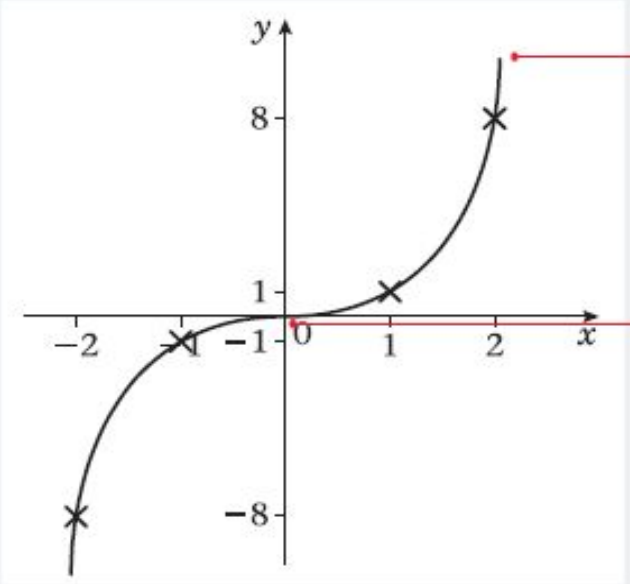
1.5.1 : Sketch the graph of cubic function given in factorized form.

Example 1 Sketch curve with equation $y=x^3$

$0 = x^3$

So the curve crosses both axes at (0, 0).

x	−2	−1	0	1	2
$y = x^3$	−8	−1	0	1	8



Put $y = 0$ and solve for x.

As the curve passes the axes at only one point, find its shape by plotting a few points.

Notice that as x increases, y increases rapidly.

The curve is 'flat' at (0, 0). This point is called a point of inflexion. The gradient is positive just before (0, 0) and positive just after (0, 0).

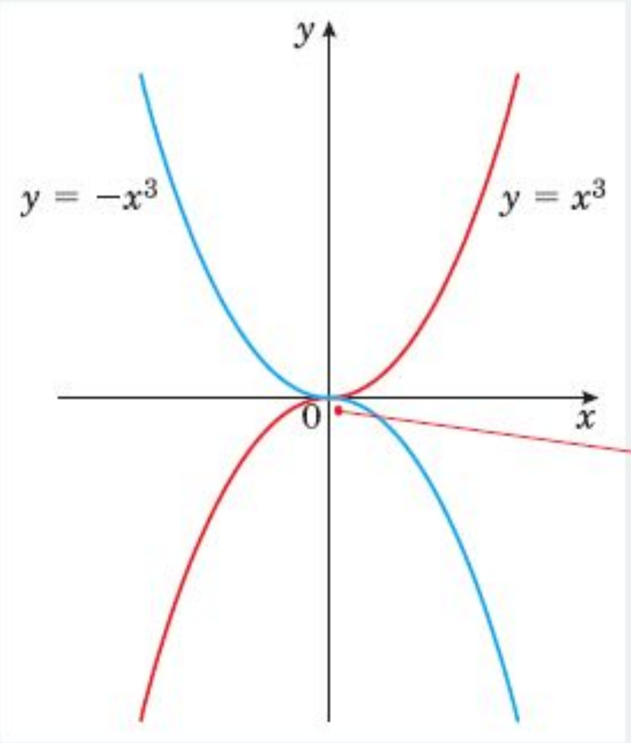
Notice that the shape of this curve is the same as the curve with equation $y = (x + 1)^3$, which is shown in Example 5.

Example 2 Sketch the curve with equations:

a. $y = -x^3$ **b.** $y = (x+1)^3$

Show their position relative to the curve with equation $y = x^3$

a $y = -x^3$



You do not need to plot any points. It is quicker if you realise the curve $y = -x^3$ is a reflection in the x-axis of the curve $y = x^3$. You can check this by looking at the values used to sketch $y = x^3$. So, for example, $x = 2$ will now correspond to $y = -8$ on the curve $y = -x^3$.

The curve is still flat at (0, 0).

b $y = (x+1)^3$

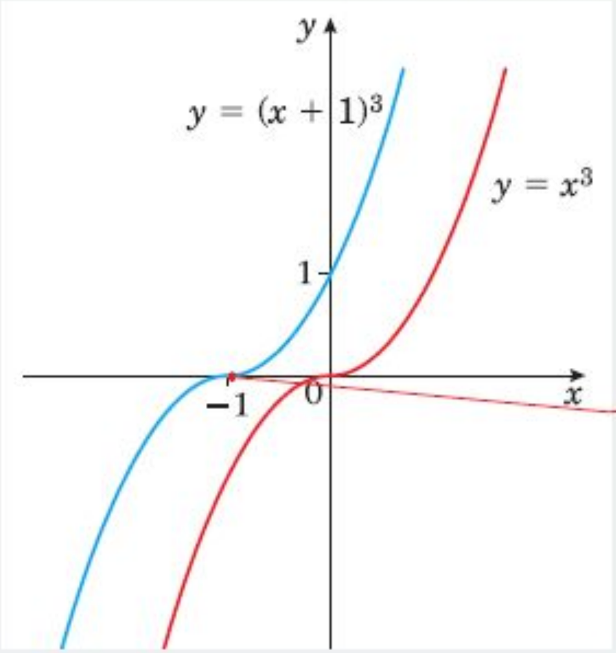
$0 = (x+1)^3$

So $x = -1$

So the curve crosses the x-axis at $(-1, 0)$.

When $x = 0$, $y = 1^3 = 1$

So the curve crosses the y-axis at $(0, 1)$.



Put $y = 0$ to find where the curve crosses the x-axis.
Put $x = 0$ to find where the curve crosses the y-axis.

The curve has the same shape as $y = x^3$.

You do not need to do any working if you realise the curve $y = (x+1)^3$ is a translation of -1 along the x-axis of the curve $y = x^3$.

The point of inflexion is at $(-1, 0)$.

Example 2A Sketch the curve with the equation $y = (x - 2)(x - 1)(x + 1)$

Put $y = 0$ and solve for x to find the roots (the points where the curve crosses the x-axis).

$0 = (x - 2)(x - 1)(x + 1)$

So $x = 2$ or $x = 1$ or $x = -1$

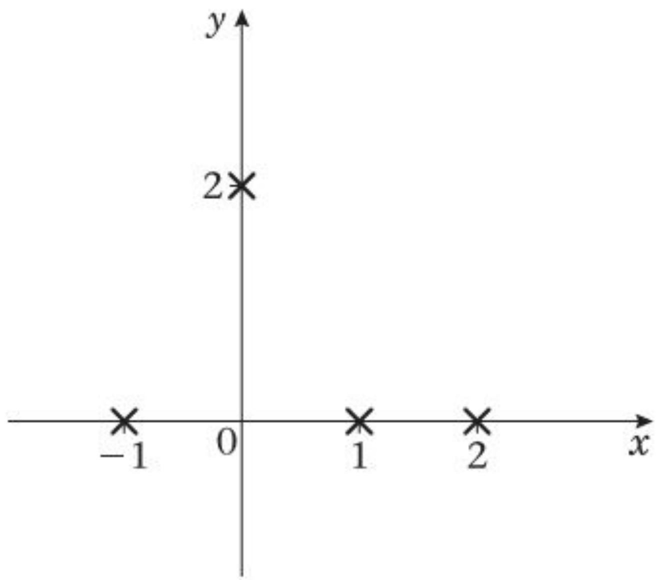
So the curve crosses the x-axis at

(2, 0) (1, 0) and (−1, 0).

When $x = 0$, $y = -2 \times -1 \times 1 = 2$

So the curve crosses the y-axis at (0, 2).

Put $x = 0$ to find where the curve crosses the y-axis.



When x is large and positive, y is large and positive.

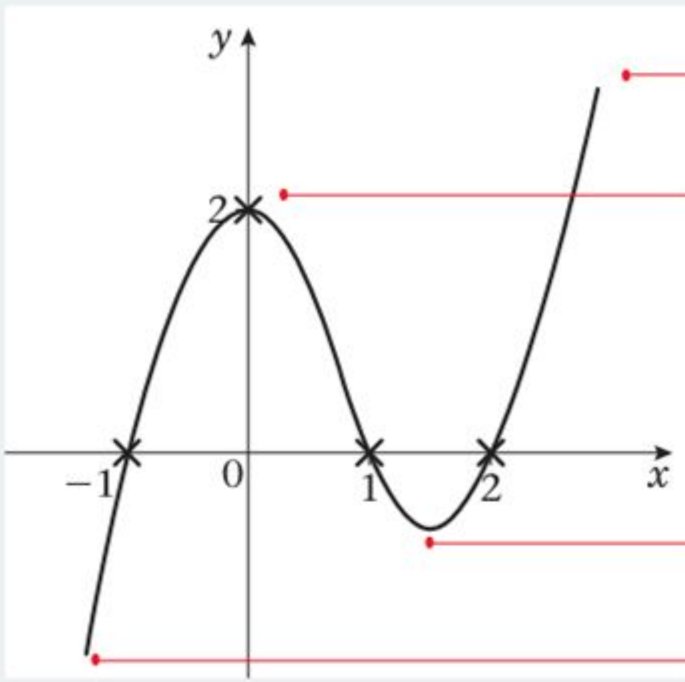
When x is large and negative, y is large and negative.

Check what happens to y for large positive and negative values of x.

You can write this as

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



$x \to \infty, y \to \infty$

This is called a local maximum because the gradient changes from +ve to 0 to −ve.

This is called a local minimum because the gradient changes from −ve to 0 to +ve.

$x \to -\infty, y \to -\infty$

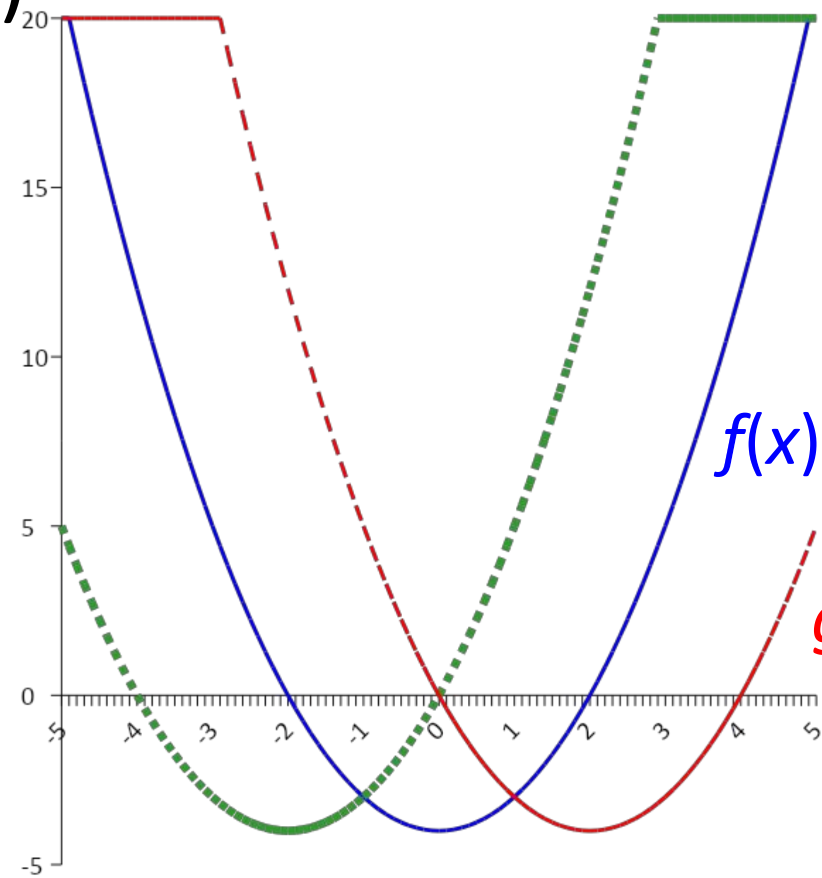
Transformations

We will now consider four elementary transformations and some simple combined transformations.

1. $f(x) \longrightarrow f(x+a)$ Translation (x-axis)
2. $f(x) \longrightarrow f(x)+a$ Translation (y-axis)
3. $f(x) \longrightarrow f(ax)$ Scaling (stretching)(x-axis)
4. $f(x) \longrightarrow af(x)$ Scaling (stretching)(y-axis)

1.5.2. Horizontal translation

$f(x) \to f(x + a)$



$f(x) = x^2 - 4$

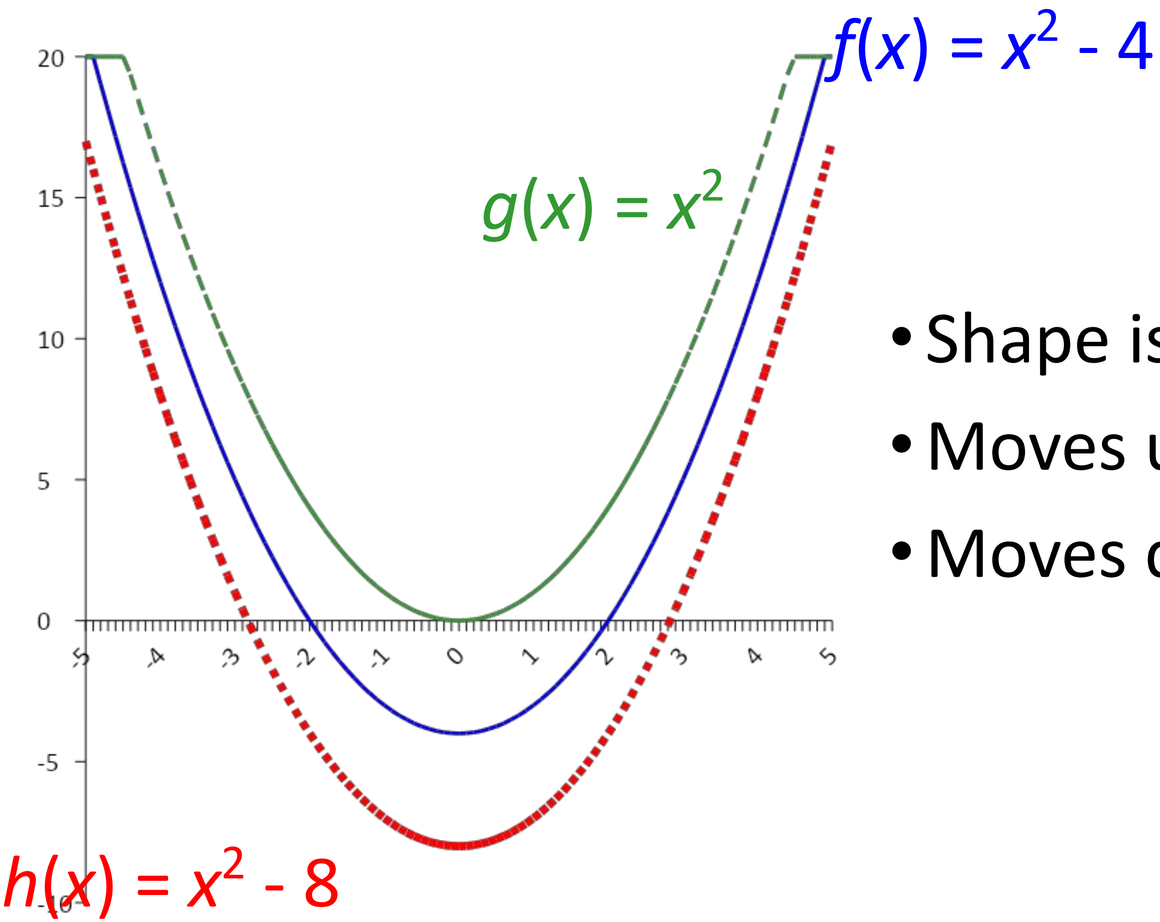
$g(x) = (x-2)^2 - 4$

$h(x) = (x+2)^2 - 4$

- Shape is unchanged.
- Moves to the left if a>0.
- Moves to the right if a<0.

1.5.3. Vertical translation

$$f(x) \rightarrow f(x) + a$$

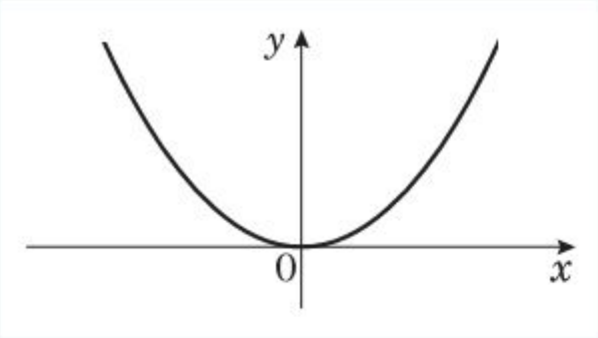


- Shape is unchanged.
- Moves upwards if a>0.
- Moves downwards if a<0.

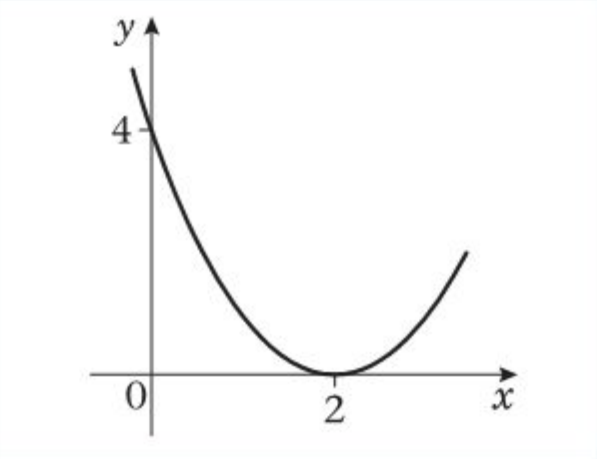
Example 3 Sketch the curves for:

a $f(x) = x^2$ **b** $g(x) = (x - 2)^2$ **c** $h(x) = x^2 + 2$

a $f(x) = x^2$

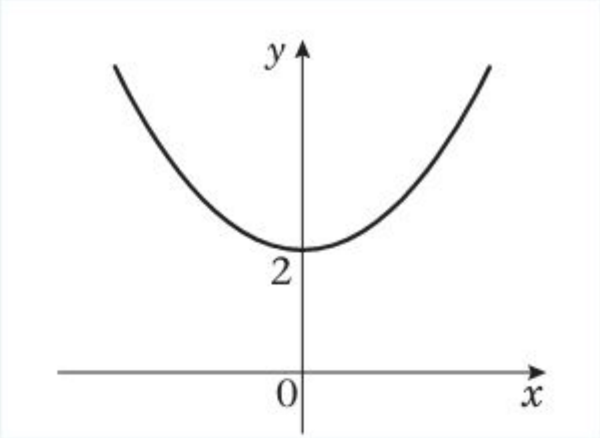


b $g(x) = (x - 2)^2$ So $g(x) = f(x - 2)$



Here $a = -2$ so $g(x)$ is a horizontal translation of $-(-2) = +2$ along the x-axis.

c $h(x) = x^2 + 2$ So $h(x) = f(x) + 2$



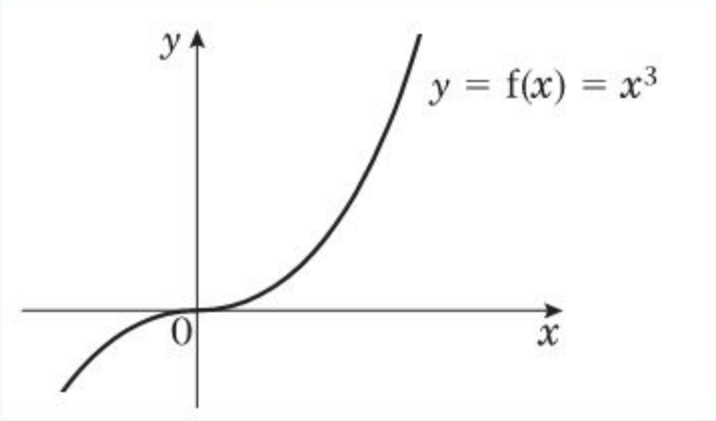
Here $a = +2$ so $h(x)$ is a vertical translation of $+2$ along the y-axis.

Example 4

a Given that **i** $f(x) = x^3$ **ii** $g(x) = x(x-2)$,
sketch the curves with equation $y = f(x+1)$ and $g(x+1)$ and mark on your sketch the points where the curves cross the axes.

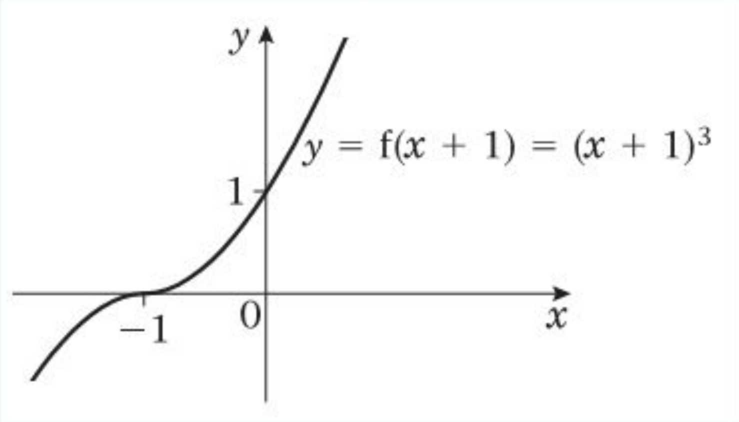
* **b** Given that $h(x) = \frac{1}{x}$, sketch the curve with equation $y = h(x) + 1$ and state the equations of any asymptotes and intersections with the axes.

a i The graph of $f(x) = x^3$ is



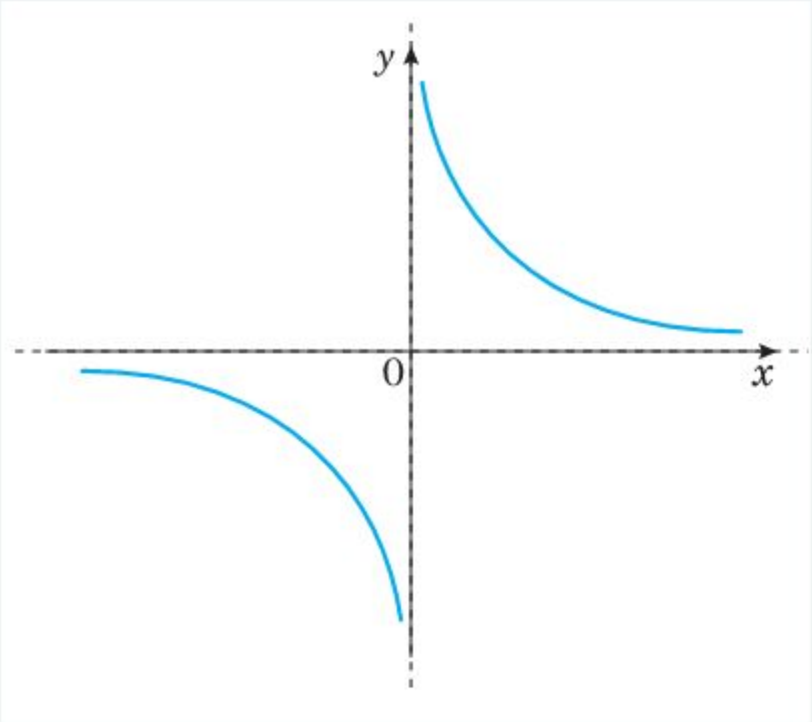
First sketch f(x).

So the graph of $y = f(x+1)$ is

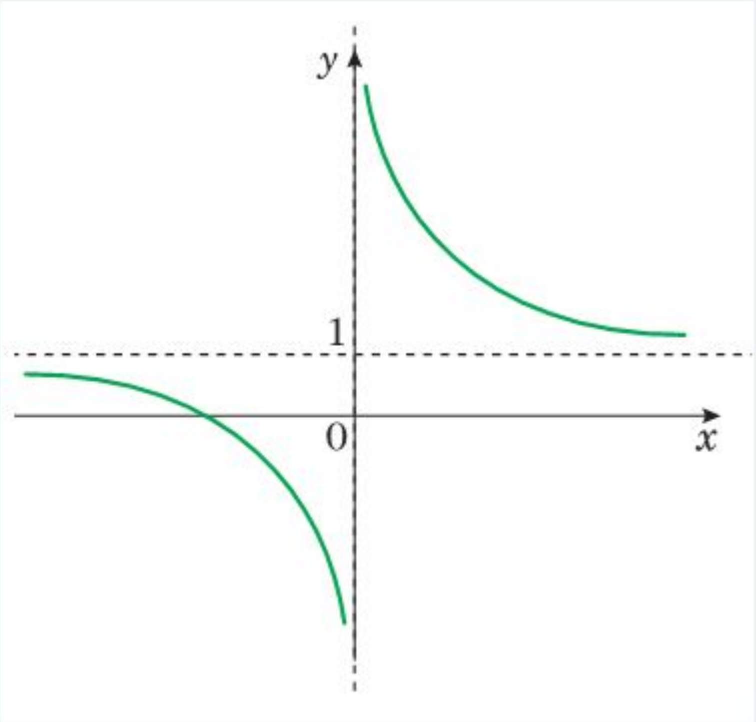


Here $a = +1$ so it is a horizontal translation of -1 along the x-axis.
In this case the new equations can easily be found as $y = (x+1)^3$ and this may help with the sketch.

*b The graph of $h(x) = \frac{1}{x}$ is



So the graph of $y = h(x) + 1$ is



First sketch h(x).

Here $a = +1$ so it is a vertical translation of $+1$ along the y-axis.

The curve crosses the x-axis once.

$$y = \mathrm{h}(x) + 1 = \frac{1}{x} + 1$$

$$0 = \frac{1}{x} + 1$$

$$-1 = \frac{1}{x}$$

$$x = -1$$

So the curve intersects the x-axis at $(-1, 0)$.

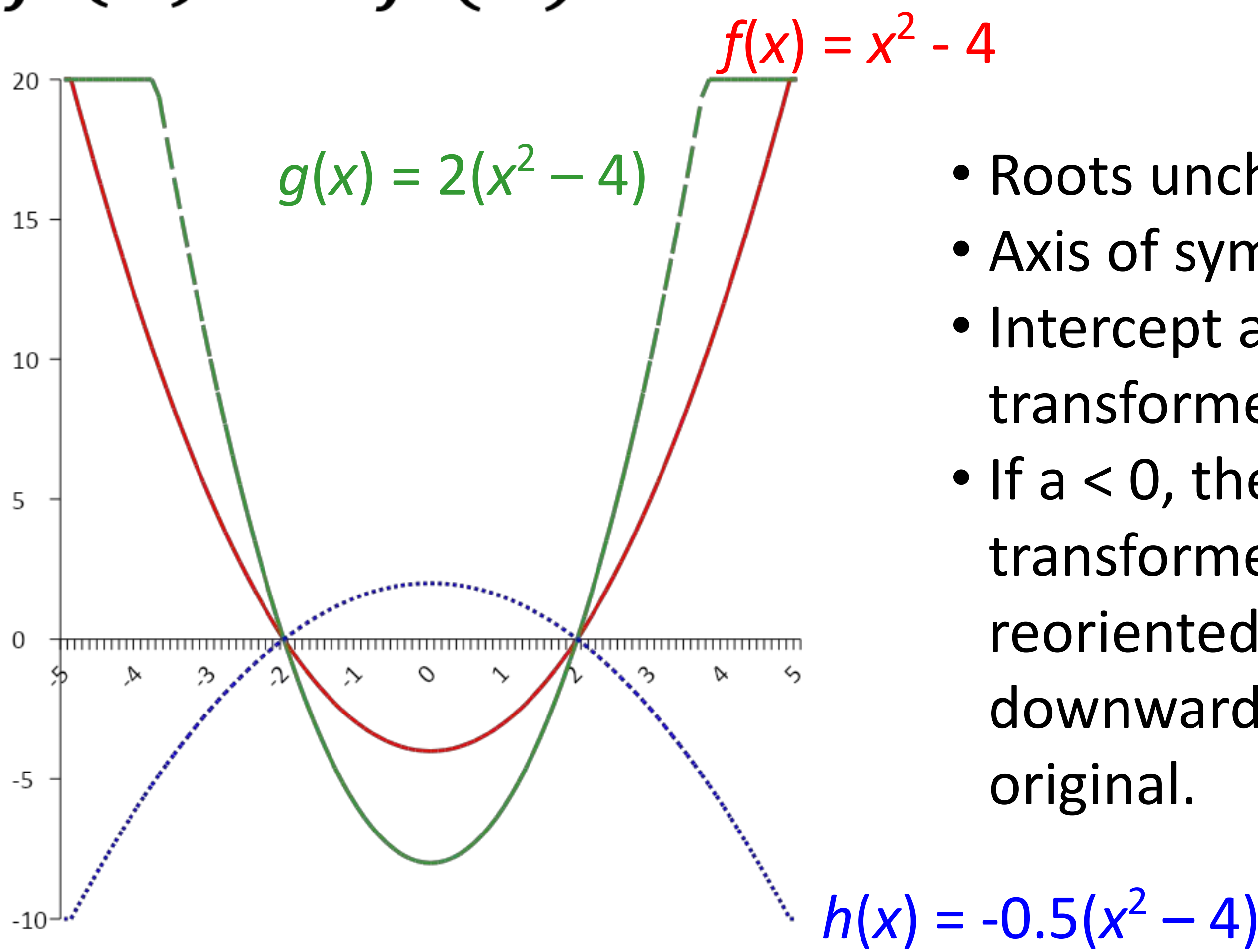
The horizontal asymptote is $y = 1$.

The vertical asymptote is $x = 0$.

Put $y = 0$ to find where the curve crosses the x-axis.

1.5.4. Vertical stretch

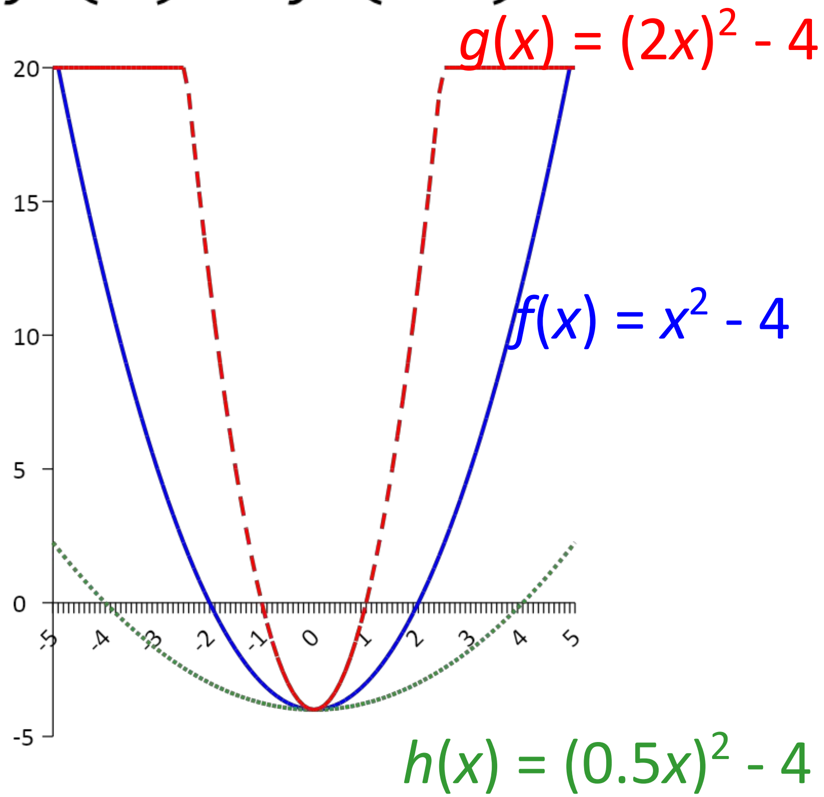
$$f(x) \rightarrow af(x)$$

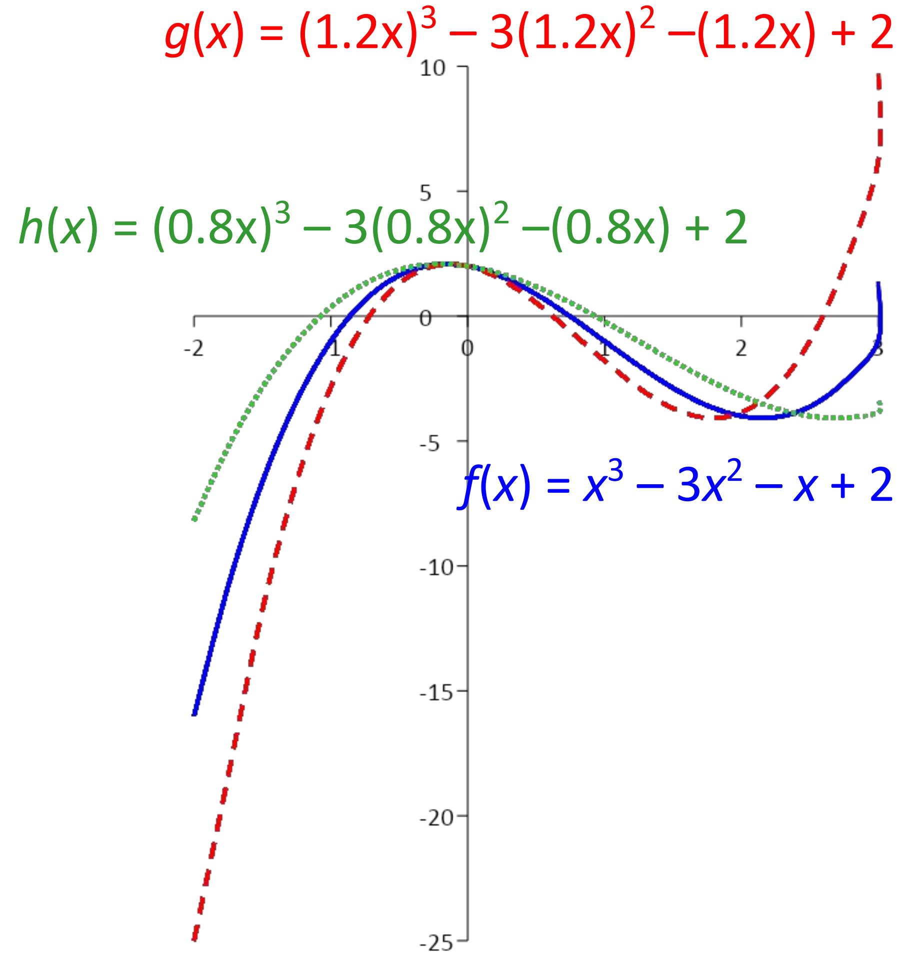


- Roots unchanged
- Axis of symmetry unchanged
- Intercept at (0, c) is transformed to (0, ac)
- If $a < 0$, the branches of transformed curve are reoriented from upward to downward or vice versa from original.

1.5.5 Horizontal stretch

$$f(x) \to f(ax)$$





Even power. Location of roots changes if |a|≠ 1. Axis of symmetry unchanged.

Odd power.

Example 5 Given that $f(x) = 9 - x^2$, sketch the curves with equations:

a $y = f(2x)$ **b** $y = 2f(x)$

a $f(x) = 9 - x^2$

So $f(x) = (3 - x)(3 + x)$

The curve is $y = (3 - x)(3 + x)$

$0 = (3 - x)(3 + x)$

So $x = 3$ or $x = -3$

So the curve crosses the x-axis at $(3, 0)$ and $(-3, 0)$.

When $x = 0$, $y = 3 \times 3 = 9$

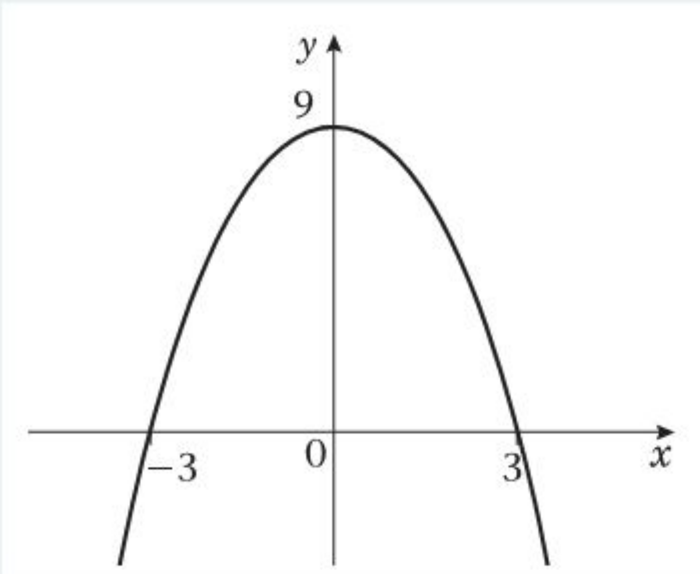
So the curve crosses the y-axis at $(0, 9)$.

You can factorise the expression.

Put $y = 0$ to find where the curve crosses the x-axis.

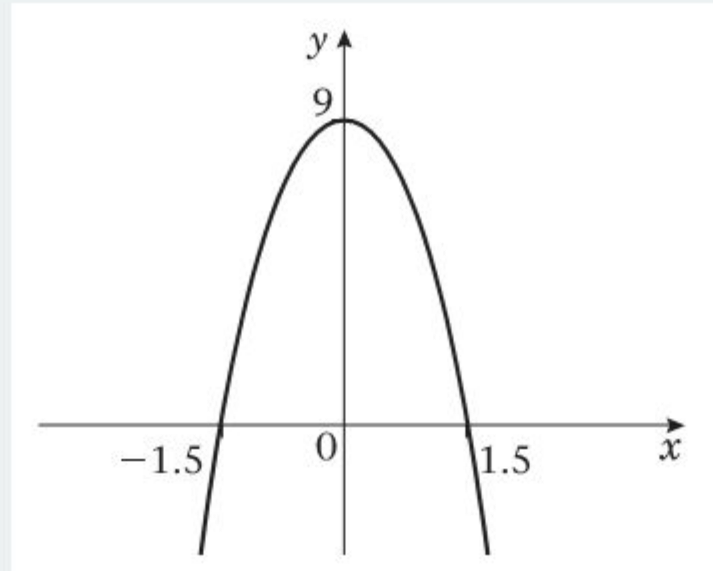
Put $x = 0$ to find where the curve crosses the y-axis.

The curve $y = \mathrm{f}(x)$ is



First sketch $y = \mathrm{f}(x)$.

$y = \mathrm{f}(2x)$ so the curve is



$y = \mathrm{f}(ax)$ where $a = 2$ so it is a horizontal stretch with scale factor $\frac{1}{2}$.

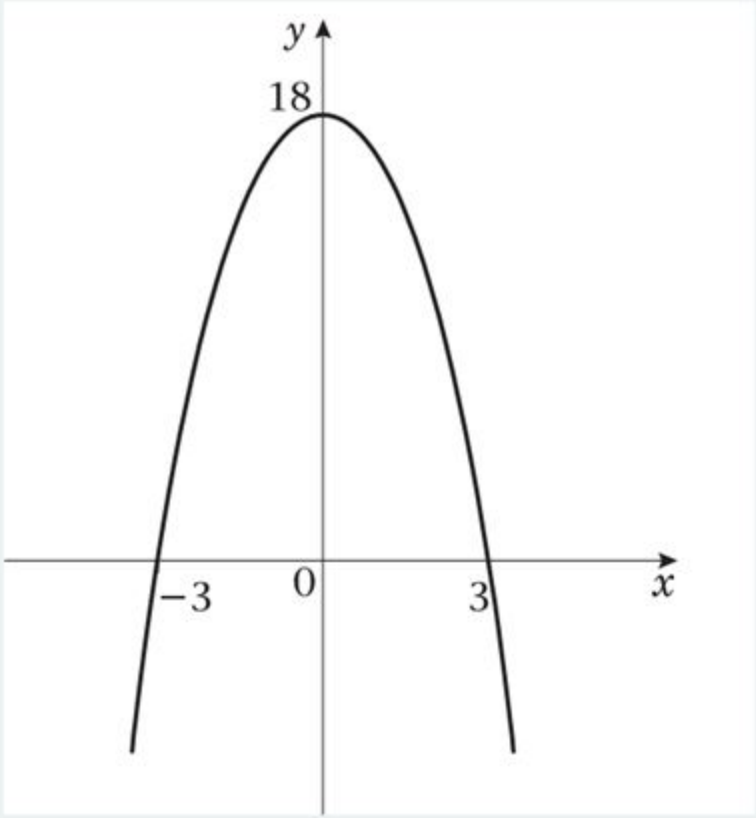
Check: The curve is $y = \mathrm{f}(2x)$. So $y = (3 - 2x)(3 + 2x)$.

When $y = 0$, $x = -1.5$ or $x = 1.5$.
So the curve crosses the x-axis at $(-1.5, 0)$ and $(1.5, 0)$.

When $x = 0$, $y = 9$.
So the curve crosses the y-axis at $(0, 9)$.

b $y = 2f(x)$

So the curve is



$y = af(x)$ where $a = 2$ so it is a vertical stretch with scale factor 2.

Check: The curve is $y = 2f(x)$.

So $y = 2(3 - x)(3 + x)$.

When $y = 0$, $x = 3$ or $x = -3$.

So the curve crosses the x-axis at $(-3, 0)$ and $(3, 0)$.

When $x = 0$, $y = 2 \times 9 = 18$.

So the curve crosses the y-axis at $(0, 18)$.

1.5.6 Simple combined transformations

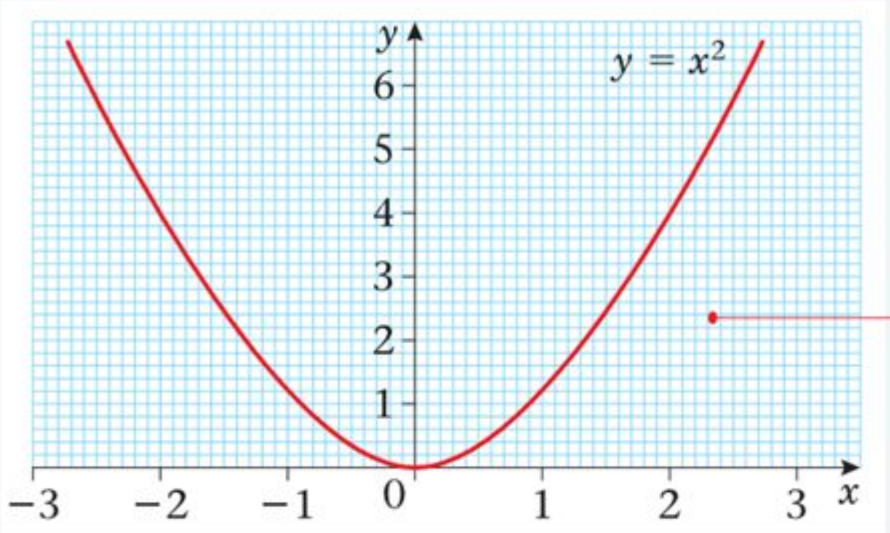
Example 6 Sketch the graph of $y = (x - 2)^2 + 3$.

Start with $f(x) = x^2$

$f(x - 2) = (x - 2)^2$

Calling this $g(x)$, $g(x) = (x - 2)^2$

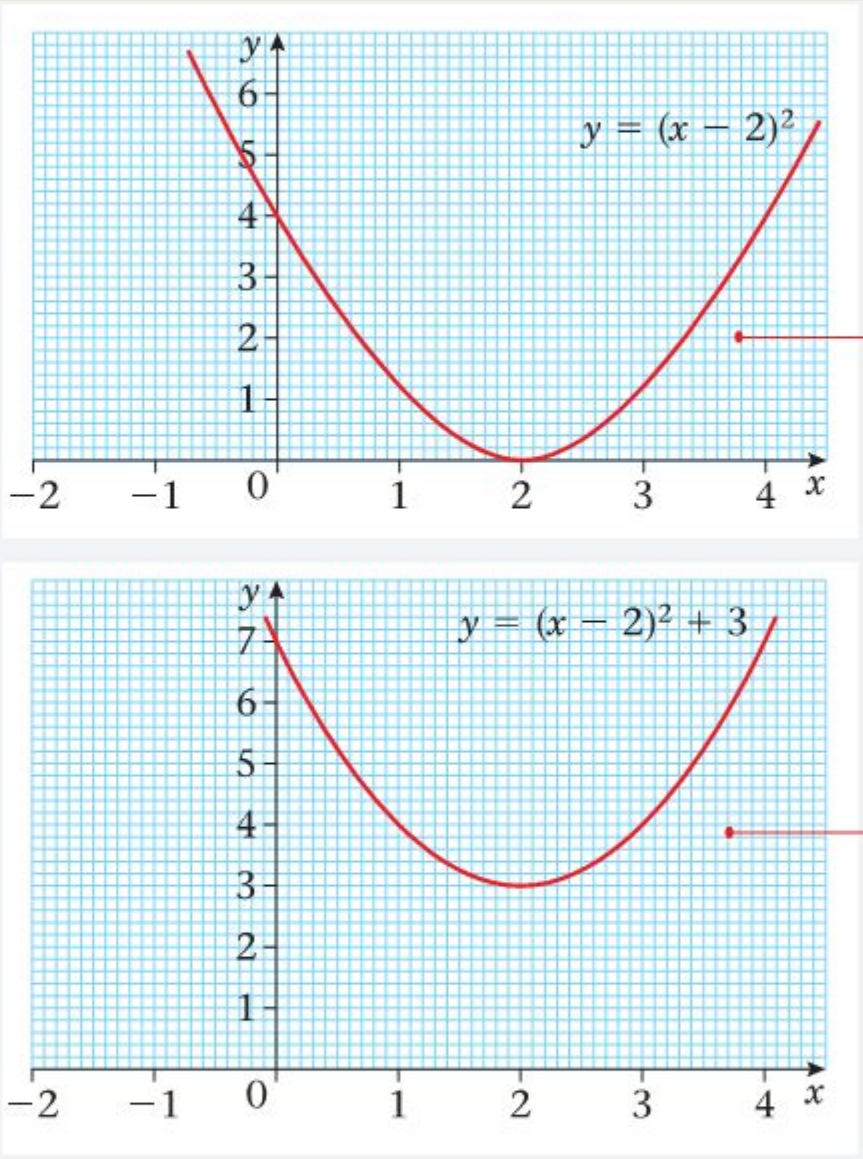
$g(x) + 3 = (x - 2)^2 + 3$



Step 1 using (1):
Horizontal translation of +2.

Step 2 using (2):
Vertical translation of +3.

Sketch the graph of $f(x) = x^2$.



Step 1
Horizontal translation of +2.

Step 2
Vertical translation of +3.

Example 7 Sketch the graph of $y = \dfrac{2}{x+5}$.

Start with $f(x) = \dfrac{1}{x}$

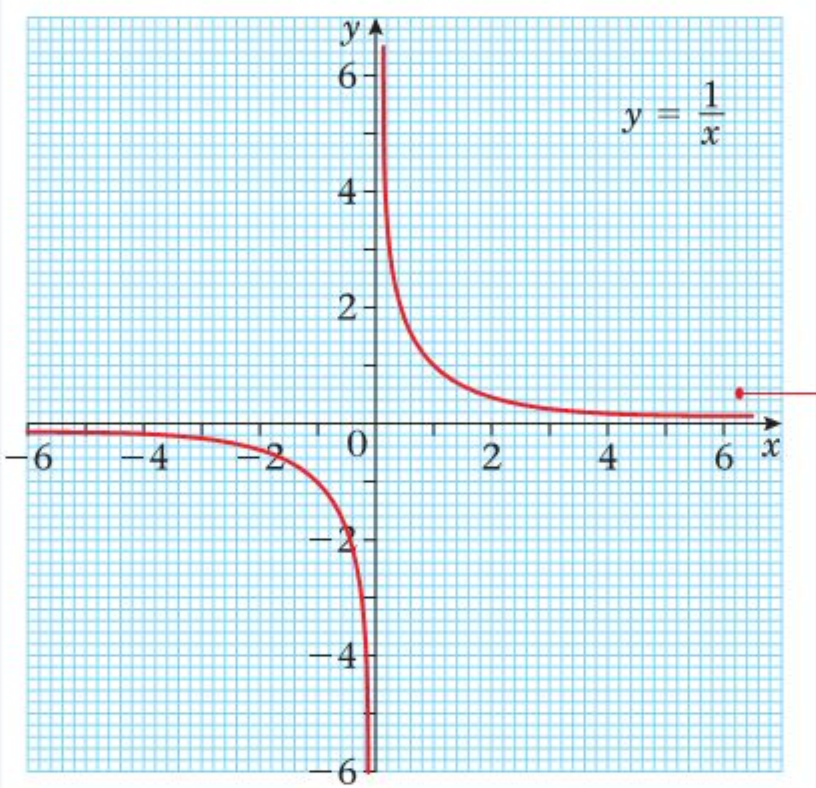
$f(x+5) = \dfrac{1}{x+5}$

Step 1 using (1):
Horizontal translation of -5

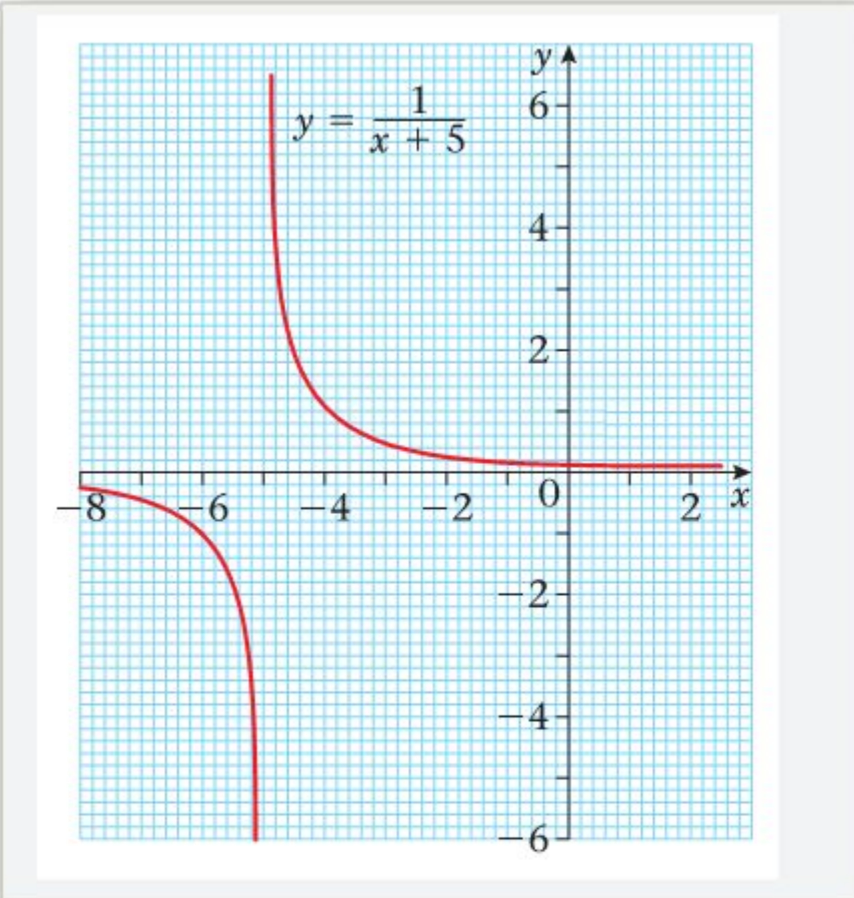
Calling this $g(x)$, $g(x) = \dfrac{1}{x+5}$

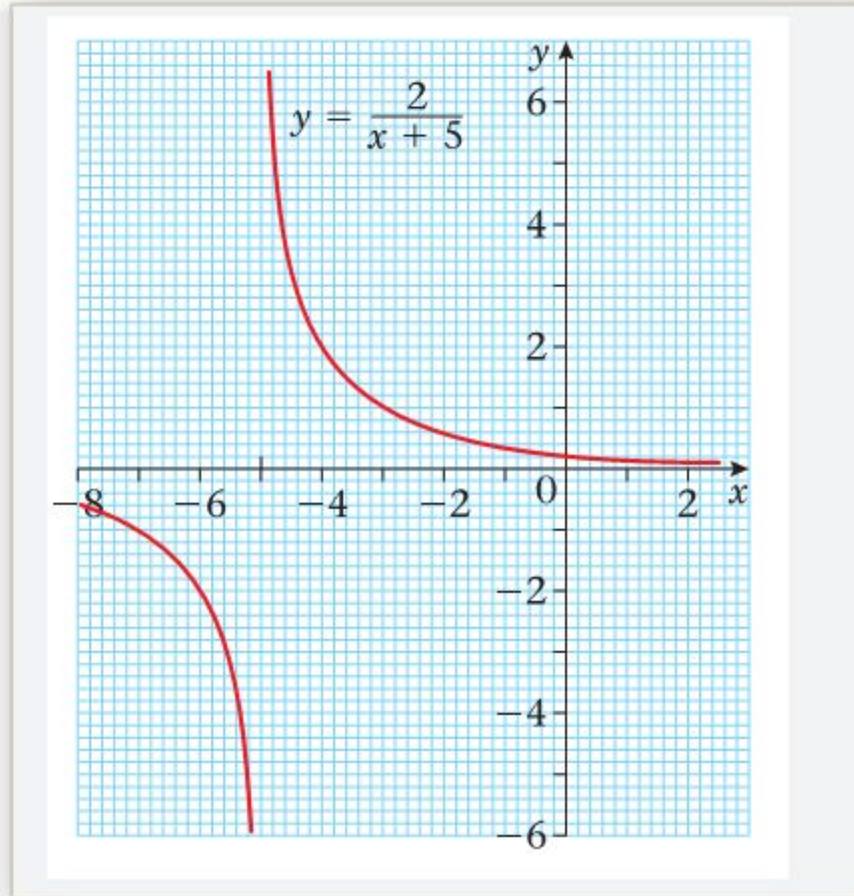
$2g(x) = \dfrac{2}{x+5}$

Step 2 using (4):
Vertical stretch, scale factor 2



Sketch the graph of $f(x) = \dfrac{1}{x}$.





Step 1
Horizontal translation of −5.

Step 2
Vertical stretch, scale factor 2.

Notice what happens to a point such as (−4, 1) .... It goes to (−4, 2).

Example 8

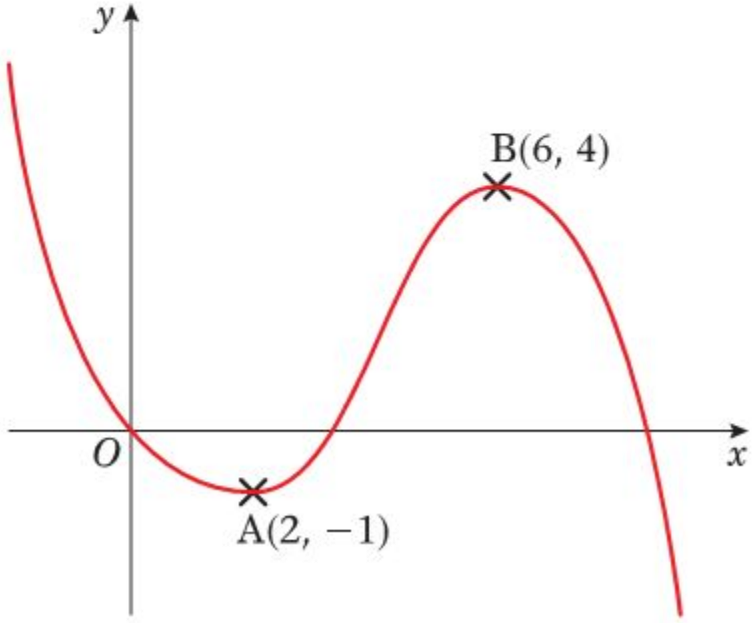
The diagram shows a sketch of the graph of $y = f(x)$. The curve passes through the origin O, the point A(2, −1) and the point B(6, 4).

Sketch the graph of:

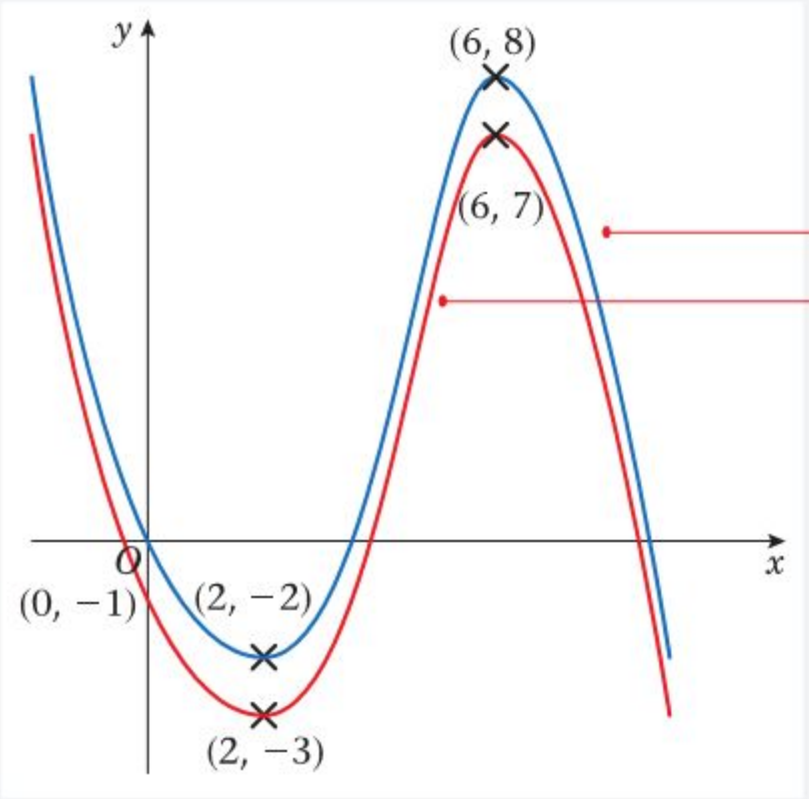
a $y = 2f(x) - 1$ **b** $y = f(x + 2) + 2$

c $y = \frac{1}{4}f(2x)$ **d** $y = -f(x - 1)$

In each case, find the coordinates of the images of the points O, A and B.



a $y = 2f(x) - 1$



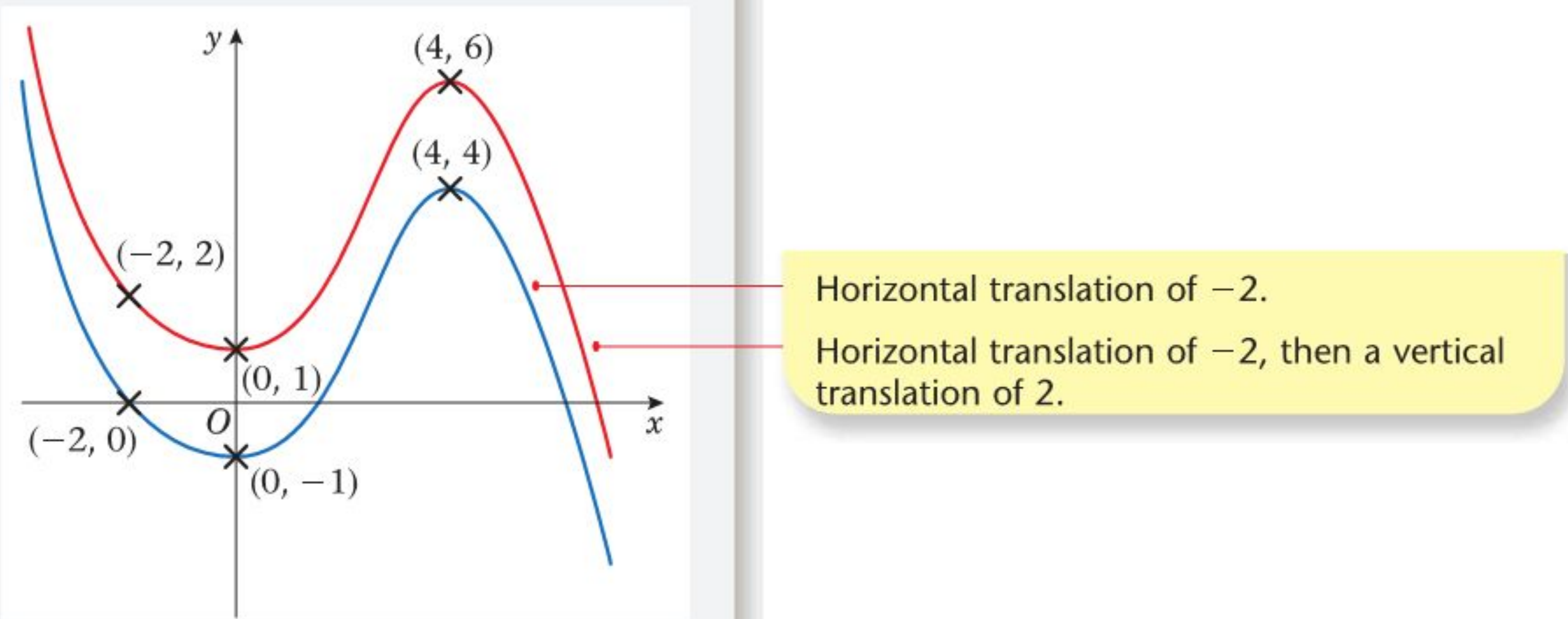
$y = 2f(x) - 1$ is shown in red in the diagram.

The images of O, A and B are $(0, -1)$, $(2, -3)$ and $(6, 7)$ respectively.

Vertical stretch, scale factor 2.

Vertical stretch, scale factor 2, then a vertical translation of −1.

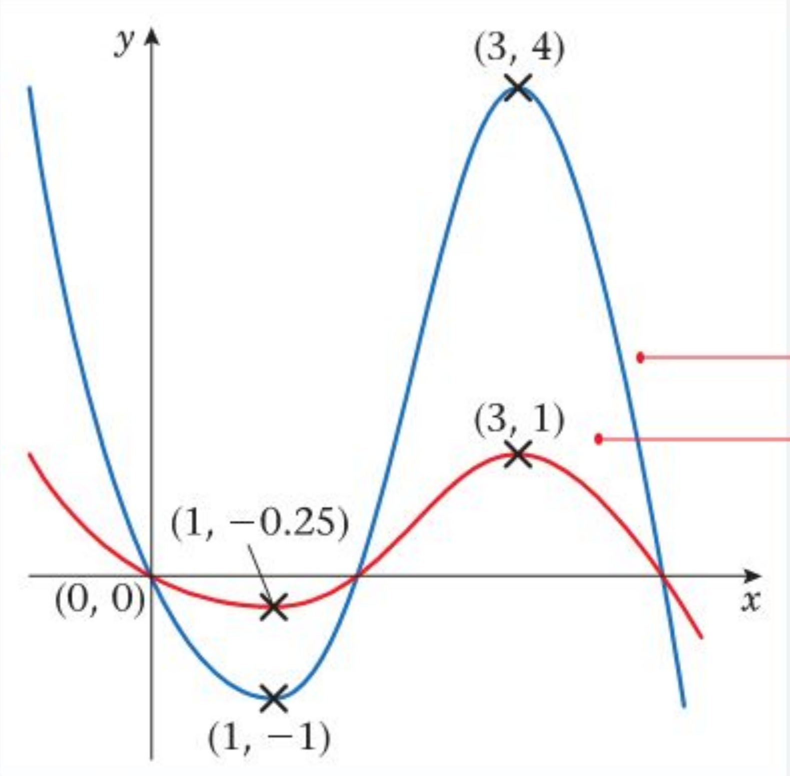
b $y = f(x + 2) + 2$



$y = f(x + 2) + 2$ is shown in red in the diagram.

The images of O, A and B are $(-2, 2)$, $(0, 1)$ and $(4, 6)$ respectively.

c $y = \frac{1}{4}f(2x)$



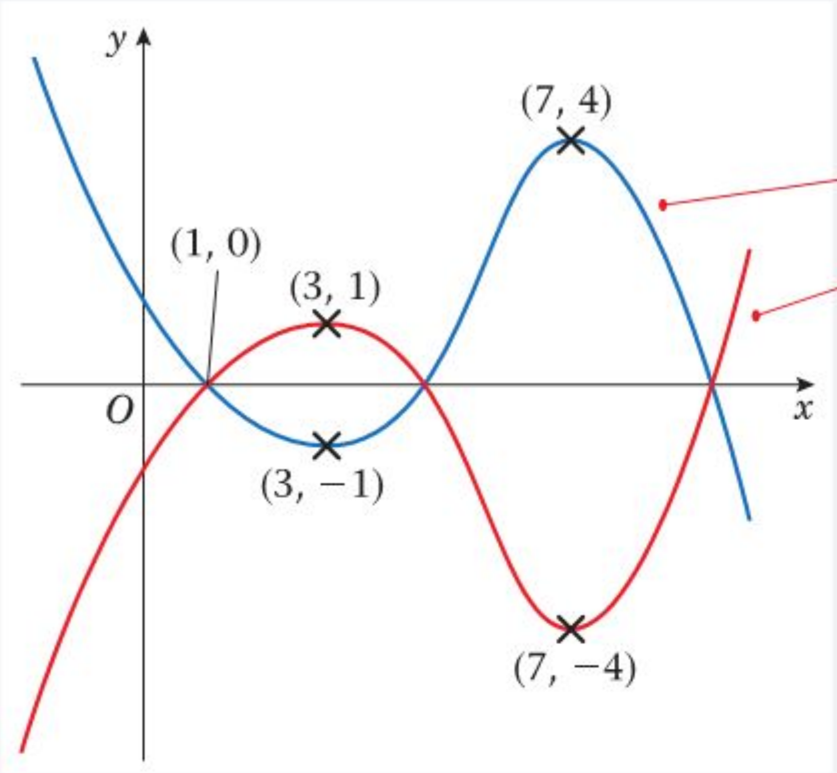
Horizontal stretch, scale factor $\frac{1}{2}$.

Horizontal stretch, scale factor $\frac{1}{2}$, then a vertical stretch, scale factor $\frac{1}{4}$.

$y = \frac{1}{4}f(2x)$ is shown in red in the diagram.

The images of O, A and B are (0, 0), (1, −0.25) and (3, 1) respectively.

d $y = -f(x-1)$



Horizontal translation of 1.

Horizontal translation of 1, then a vertical stretch, scale factor −1.

A 'vertical stretch with scale factor −1' is equivalent to a reflection in the x-axis.

$y = -f(x-1)$ is shown in red in the diagram.

The images of O, A and B are (1, 0), (3, 1) and (7, −4) respectively.

In this lecture, we covered

1.5.1 Sketch the graph of a cubic function given in factorized form.

1.5.2 Apply a horizontal translation to a given curve.

1.5.3 Apply a vertical translation to a given curve.

1.5.4 Apply a vertical stretch to a given curve.

1.5.5 Apply a horizontal stretch to a given curve.

1.5.6 Apply simple combined transformations to a given curve.

Next lecture

1.6 Simultaneous equations

Material in some of these slides has been reproduced from:

1) Attwood, G., Macpherson, A., Moran, B., Petran, J., Pledger, K., Staley, G. and Wilkins, D. (2008), Edexcel AS and A Level Modular Mathematics series C1, Pearson, Harlow, UK.

2) Attwood, G., Macpherson, A., Moran, B., Petran, J., Pledger, K., Staley, G. and Wilkins, D. (2008), Edexcel AS and A Level Modular Mathematics series C3, Pearson, Harlow, UK.