“Newton’s binomial formula ”

Newton’s formula

There is: Binomial’s theorem , a,bє Ł si nєAt*, then

$$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \ldots + C_n^k a^{n-k} b^k + \ldots + C_n^{n-1} a b^{n-1} + C_n^n b^n$$

known also as Newton’s formula.
Isaac Newton, English mathematician, astronomer, physician (1643-1727)

Demonstration using mathematic induction method:
Step I. Verification : P(1): ……. Independent work …..

Theorem demonstration :

Fie $P(n):(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1}b + C_n^2 a^{n-2}b^2 + \ldots + C_n^k a^{n-k}b^k + \ldots + C_n^n b^n, n \in \mathbb{N}$.

I. Verificare: $P(1): \quad (a+b)^1 = C_1^0 a + C_1^1 b$ **(A)**;

II. $P(n) \rightarrow P(n+1)$:

$P(n+1):(a+b)^{n+1} = C_{n+1}^0 a^{n+1} + C_{n+1}^1 a^n b + \ldots C_{n+1}^k a^{n+1-k} b^k + \ldots + C_{n+1}^{n+1} b^{n+1}$ **(?)**

$P(n+1):(a+b)(a+b)^n = (a+b)\left(C_n^0 a^n + C_n^1 a^{n-1}b + \ldots + C_n^k a^{n-k}b^k + \ldots + C_n^n b^n\right) =$

$= C_n^0 a^{n+1} + C_n^1 a^n b + \ldots + C_n^k a^{n-k+1} b^k + \ldots + C_n^n b^n + C_n^0 a^n b + C_n^1 a^{n-1} b^2 + \ldots +$

$+ C_n^k a^{n-k} b^{k+1} + \ldots + C_n^n b^{n+1} \Rightarrow$

$(a+b)^{n+1} = \underbrace{C_n^0}_{C_{n+1}^0} a^{n+1} + \underbrace{\left(C_n^1 + C_n^0\right)}_{C_{n+1}^1} a^n b + \underbrace{\left(C_n^2 + C_n^1\right)}_{C_{n+1}^2} a^{n-1} b^2 + \ldots + \underbrace{C_n^n}_{C_{n+1}^{n+1}} C_n^n b^{n+1}$ **(A)**.

Specifications regarding Newton's formula:

1.the coefficients $C_n^0, \ldots, C_n^k$ se humes C_n^n are called binomial coefficients of the development and are in number of n+1.

Is necessary to make a distinction between the binomial coefficient of a term and the numerical coefficient of the same term.

2. Those n+1 are

$$T_1 = C_n^0 a^n,\ T_2 = C_n^1 a^{n-1} b,\ T_3 = C_n^2 a^{n-2} b^2, \ldots, \boxed{T_{k+1} = C_n^k a^{n-k} b^k}, \ldots, T_{n+1} = C_n^n b^n.$$

3. The natural numbers $C_n^0, C_n^2, C_n^4 \ldots$ are called binomial coefficients of odd rank, and the numbers $C_n^1, C_n^3, C_n^5 \ldots$ are called binomial coefficients of even rank.

4. In Newton's formula the exponents of a powers are decreasing from n to 0, and exponents of b power are increasing from 0 to n.

Specifications regarding Newton's formula (continuation)

5. The binomial coefficients of the extreme terms and those equally distant from the extreme terms are equal : $C_n^0=C_n^n,\ \ C_n^1=C_n^{n-1},\ \ C_n^2=C_n^{n-2},\ ....,\ C_n^k=C_n^{n-k}.$

6. If the power exponent is even, n=2k, then the development has 2k+1 terms, and the middle term has the highest binominal coefficient :

$$C_n^0 < C_n^1 < C_n^2 .... < C_n^k > C_n^{k+1} > .... > C_n^n.$$

If the power exponent is odd, n=2k+1, then the development has 2k+2 terms and there are two terms in the middle of the development with equally binomial coefficients and of highest value

$$C_n^0 < C_n^1 < C_n^2 .... < C_n^k = C_n^{k+1} > C_n^{k+2} .... > C_n^n.$$

7. An important role, in resolving problems related with Newton's binomial, is played by the **general term** having the rank k+1:

$$T_{k+1} = C_n^k a^{n-k} b^k,\ k\ \{\ \}$$

Example:

1 $(1+2x)^6 = 1 + C_6^1 \cdot 2x + C_6^2(2x)^2 + C_6^3(2x)^3 + C_6^4(2x)^4 + C_6^5(2x)^5 + C_6^6(2x)^6$

Thus:

a) $T_4 = C_6^3(2x)^3 = 160x^3$

b) The binomial coefficient of T_3 is $C_6^2 = 15$

c) The coefficient of T_5 is $C_6^4 \cdot 2^4 = 240$

d) The free term $T_1 = 1$

e) The term that contain ^{5} is ${}_{6} = {}^{5}_{6}(\quad)^{5} = \quad x^{5}$

f) there is no term that contains ^{9}

Identities in the combination calculus

Using the Newton's formula for binomial development $(a+b)^n$

$$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \ldots . C_n^k a^{n-k} b^k + \ldots . + C_n^{n-1} a b^{n-1} + C_n^n b^n,$$

There can be deduced some interesting identities in which binomial coefficients intervene.

- Particularised in Newton's formula a=b=1 we find :

$$2^n = C_n^0 + C_n^1 + C_n^2 + \ldots . + C_n^{n-1} + C_n^n$$

the sum of the development of the binomial coefficients is 2^n

- In the same formula taking a=1 and b=-1 we obtain:

$$0 = C_n^0 - C_n^1 + C_n^2 - \ldots . (-1)^n C_n^n$$

the alternating sum of the binomial coefficients is 0

Identities in the combination calculus(continuation)

Adding the two sums member by member we obtain:

$$
\begin{array}{l}
2^n = C_n^0 + C_n^1 + C_n^2 + \ldots\ldots + C_n^{n-1} + C_n^n \\
0 = C_n^0 - C_n^1 + C_n^2 - \ldots\ldots (-1)^n C_n^n \\
\hline
2^n = 2\left(C_n^0 + C_n^2 + C_n^4 + C_n^6 + \ldots\ldots\right)
\end{array}
$$

Or : $2^{n-1} = C_n^0 + C_n^2 + C_n^4 + C_n^6 + \ldots.$

the sum of the binomial coefficients of odd rank is 2^{n-1}

Subtracting the two sum we obtain

$$2^n = 2\left(C_n^1 + C_n^3 + C_n^5 + C_n^7 + \ldots\ldots\right)$$ or

$$2^{n-1} = C_n^1 + C_n^3 + C_n^5 + C_n^7 + \ldots.$$

The sum of the binomial coefficients of even rank is 2^{n-1}

Aplication:

6. Calculate the sum :

$$S_n = C_n^1 + 2C_n^2 + 3C_n^3 + \ldots.. + nC_n^n .$$

a) using the equality $kC_n^k = nC_{n-1}^{k-1}$

for n,k є Ł and n ≥ k

b) using the complementary combination’s formula

$C_n^k = C_n^{n-k}$ for n,k є Ł and n ≥ k

Answer:

a) demonstration of the formula

$$kC_n^k = k\frac{n!}{k!.(n-k)!} = k\frac{n(n-1)!}{k(k-1)!(n-k)!} =$$

$$= n\frac{(n-1)!}{(k-1)!(n-k)!} = nC_{n-1}^{k-1}$$

Thus the sum is rewritten

$$S_n = nC_{n-1}^0 + nC_{n-1}^1 + nC_{n-1}^2 + ... + nC_{n-1}^{n-1} =$$

$$= n\left(C_{n-1}^0 + C_{n-1}^1 + C_{n-1}^2 + ... + C_{n-1}^{n-1}\right) = n \cdot 2^{n-1}$$

Test

It is given the binomial : $\left(\frac{1}{x}-\sqrt[3]{x}\right)^{14}, x \neq 0.$

1. How many terms does the development has?
2. Which is the rank of the middle term?
3. Which is the sum of the binomial coefficients of this binomial? Using the general term formula, $T_{k+1}=C_n^k a^{n-k} b^k$ find out :
4. The rank of the term that contains x^2.
5. How many rational terms does the development has?