Solution Methods for Bilevel Optimization

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Overview

- Definition and general form of a bilevel problem
- Discuss optimality (KKT-type) conditions
- Reformulate general bilevel problem as a system of equations
- Consider iterative (descent direction) methods applicable to solve this reformulation
- Look at the numerical results of using Levenberg-Marquardt method

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Stackelberg Game (Bilevel problem)

- Players: the Leader and the Follower
- The Leader is first to make a decision
- Follower reacts optimally to Leader’s decision
- The payoff for the Leader depends on the follower’s reaction

Example

- Taxation of a factory
- Leader – government
- Objectives: maximize profit and minimize pollution
- Follower – factory owner
- Objectives: maximize profit

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General structure of a Bilevel problem

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y$$

subject to $A_1 x + B_1 y \leq b_1$

$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y$$

subject to $A_2 x + B_2 y \leq b_2$

Important Sets

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$$\min_{x \in X} F(x, y) = c_1 x + d_1 y$$

$$\text{subject to} \quad A_1 x + B_1 y \leq b_1$$

$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y$$

$$\text{subject to} \quad A_2 x + B_2 y \leq b_2$$

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Solution methods

- Vertex enumeration in the context of Simplex method
- Kuhn-Tucker approach
- Penalty approach
- Extract gradient information from a lower objective function to compute directional derivatives of an upper objective function

Concept of KKT conditions

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y$$

subject to $A_1 x + B_1 y \leq b_1$

$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y$$

subject to $A_2 x + B_2 y \leq b_2$

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Value function reformulation

$$\min_{x \in X} F(x,y) = c_1 x + d_1 y$$

subject to $A_1 x + B_1 y \leq b_1$

$$\min_{y \in Y} f(x,y) = c_2 x + d_2 y$$

subject to $A_2 x + B_2 y \leq b_2$

KKT for value function reformulation

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y$$

$$\text{subject to} \quad A_1 x + B_1 y \leq b_1$$

$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y$$

$$\text{subject to} \quad A_2 x + B_2 y \leq b_2$$

Assumptions

KKT-type optimality conditions for B

$$
\begin{aligned}
\nabla_x F(\bar{x},\bar{y}) + r\nabla_x f(\bar{x},\bar{y}) - r\sum_{s=1}^{n+1}\eta_s \nabla_x f(\bar{x},y_s) & \\
+\nabla_x g(\bar{x},\bar{y})^T u - r\sum_{s=1}^{n+1}\eta_s \nabla_x g(\bar{x},y_s)^T u_s & \\
+\nabla_x h(\bar{x},\bar{y})^T v - r\sum_{s=1}^{n+1}\eta_s \nabla_x h(\bar{x},y_s)^T v_s & \\
+\nabla G(\bar{x})^T u' + \nabla H(\bar{x})^T v' &= 0, \\
\nabla_y F(\bar{x},\bar{y}) + r\nabla_y f(\bar{x},\bar{y}) + \nabla_y g(\bar{x},\bar{y})^T u + \nabla_y h(\bar{x},\bar{y})^T v &= 0, \\
\nabla_y f(\bar{x},y_s) + \nabla_y g(\bar{x},y_s)^T u_s + \nabla_y h(\bar{x},y_s)^T v_s &= 0, \\
u \geq 0,\ u^T g(\bar{x},\bar{y}) &= 0, \\
u' \geq 0,\ u'^T G(\bar{x}) &= 0, \\
u_s \geq 0,\ u_s^T g(\bar{x},y_s) &= 0.
\end{aligned}
$$

Further Assumptions (for simpler version)

Simpler version

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y$$

subject to $A_1 x + B_1 y \leq b_1$

$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y$$

subject to $A_2 x + B_2 y \leq b_2$

NCP-Functions

- Define
- Give a reason (non-differentiability of constraints)
- Fischer-Burmeister

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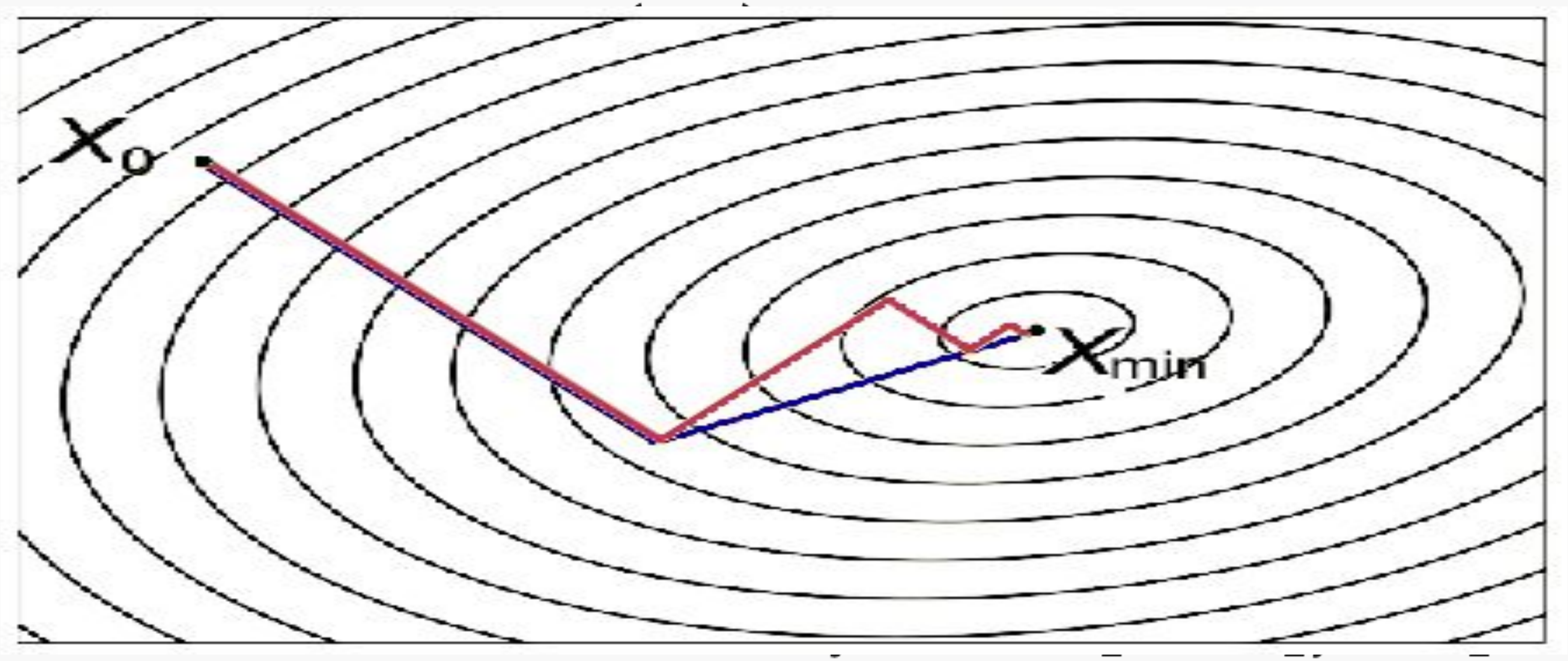
Simpler version in the form of the system of equations

$$B(z) = \begin{pmatrix} \nabla_x F(\bar{x},\bar{y}) + \nabla_x g(\bar{x},\bar{y})^T u - r\eta_s \nabla_x g(\bar{x},\bar{y})^T u_s + \nabla G(\bar{x})^T u' \\ \nabla_y F(\bar{x},\bar{y}) + \nabla_y g(\bar{x},\bar{y})^T (u - r u_s) \\ \nabla_y f(\bar{x},\bar{y}) + \nabla_y g(\bar{x},\bar{y})^T u_s \\ \sqrt{u^2 + g(\bar{x},\bar{y})^2 + \mu} - u + g(\bar{x},\bar{y}) \\ \sqrt{u'^2 + G(\bar{x})^2 + \mu} - u' + G(\bar{x}) \\ \sqrt{u_s^2 + g(\bar{x},\bar{y})^2 + \mu} - u_s + g(\bar{x},\bar{y}) \end{pmatrix} = 0,$$

Iterative methods



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For Bilevel case

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y$$

$$\text{subject to} \quad A_1 x + B_1 y \le b_1$$

$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y$$

$$\text{subject to} \quad A_2 x + B_2 y \le b_2$$

Newton method

- Define
- Explain that we are dealing with non-square system
- Suggest pseudo inverse Newton

Pseudo inverse

Newton method with pseudo inverse

Gauss-Newton method

- Define
- Mention the wrong formulation
- Refer to pseudo-inverse Newton

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Gauss-Newton method

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y$$

subject to $A_1 x + B_1 y \leq b_1$

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subject to $A_2 x + B_2 y \leq b_2$

Convergence of Newton and Gauss-Newton

- Talk about starting point condition
- Interest for future analysis

Levenberg-Marquardt method

Numerical results

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Plans for further work

$$\min_{x \in X} F(x,y) = c_1 x + d_1 y$$

subject to $A_1 x + B_1 y \leq b_1$

$$\min_{y \in Y} f(x,y) = c_2 x + d_2 y$$

subject to $A_2 x + B_2 y \leq b_2$

Plans for further work

6. Construct the own code for Levenberg-Marquardt method in the context of solving bilevel problems within defined reformulation.

7. Search for good starting point techniques for our problem. 8. Do the numerical calculations for the harder reformulation defined .

9. Code Newton method with pseudo-inverse.

10. Solve the problem assuming strict complementarity

11. Look at other solution methods.

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References

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References

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