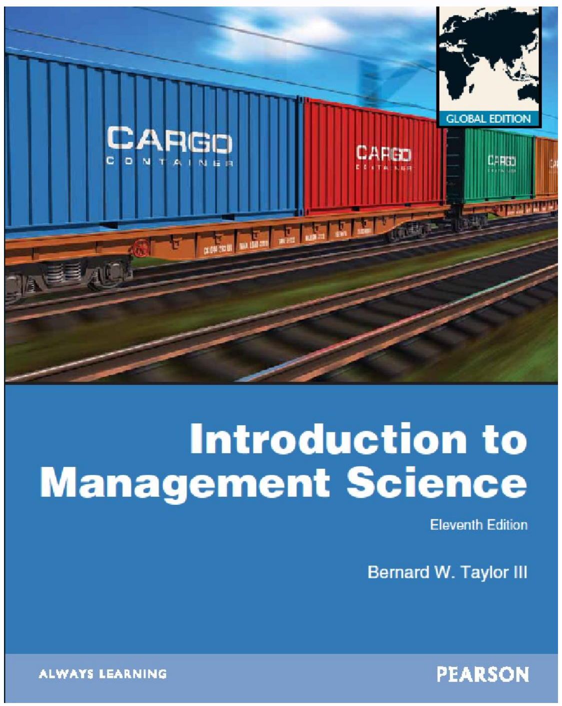
Probabilistic Models



Chapter 11

Chapter Topics

- Types of Probability
- Fundamentals of Probability
- Statistical Independence and Dependence
- Expected Value
- The Normal Distribution

Overview

- ***Deterministic*** techniques assume that no uncertainty exists in model parameters. Chapters 2-10 introduced topics that are not subject to uncertainty or variation.
- ***Probabilistic*** techniques include uncertainty and assume that there can be more than one model solution.
  - There is some doubt about which outcome will occur.
  - Solutions may be in the form of averages.

Types of Probability
Objective Probability

- ***Classical, or a priori (prior to the occurrence)***, probability is an objective probability that can be stated prior to the occurrence of the event. It is based on the logic of the process producing the outcomes.
- ***Objective probabilities*** that are ***stated after the outcomes*** of an event have been observed are ***relative frequencies***, based on observation of past occurrences.
- ***Relative frequency*** is the more widely used definition of ***objective probability.***

Types of Probability
Subjective Probability

- ***Subjective probability*** is an estimate based on personal belief, experience, or knowledge of a situation.
- It is ***often the only means available*** for making probabilistic estimates.
- Frequently used in making business decisions.
- Different people often arrive at different subjective probabilities.
- ***Objective probabilities*** are used in this text ***unless otherwise indicated.***

Fundamentals of Probability Outcomes and Events

- An *experiment* is an activity that results in one of several possible *outcomes which are termed events.*
- The *probability* of an event is always *greater than or equal to zero and less than or equal to one.*
- The *probabilities of all the events* included in an experiment must *sum to one.*
- The events in an experiment are *mutually exclusive if only one can occur at a time.*
- The *probabilities of mutually exclusive events sum to one*.

Fundamentals of Probability Distributions

- A ***frequency distribution*** is an organization of numerical data about the events in an experiment.

- A list of ***corresponding probabilities for each event*** is referred to as a ***probability distribution*.**

- A set of events is ***collectively exhaustive when it includes all the events*** that can occur in an experiment.

Fundamentals of Probability
A Frequency Distribution Example

State University, 3000 students, management science grades for past four years.

Event Grade	Number of Students	Relative Frequency	Probability
A	300	300/3,000	.10
B	600	600/3,000	.20
C	1,500	1,500/3,000	.50
D	450	450/3,000	.15
F	150	150/3,000	.05
	3,000		1.00

Fundamentals of Probability
Mutually Exclusive Events & Marginal Probability

- A ***marginal probability*** is the probability of a single event occurring, denoted by P(A).
- For mutually exclusive events, the probability that one or the other of several events will occur is found by summing the individual probabilities of the events:

$$P(A \text{ or } B) = P(A) + P(B)$$

- A ***Venn diagram*** is used to show mutually exclusive events.

Fundamentals of Probability
Mutually Exclusive Events & Marginal Probability

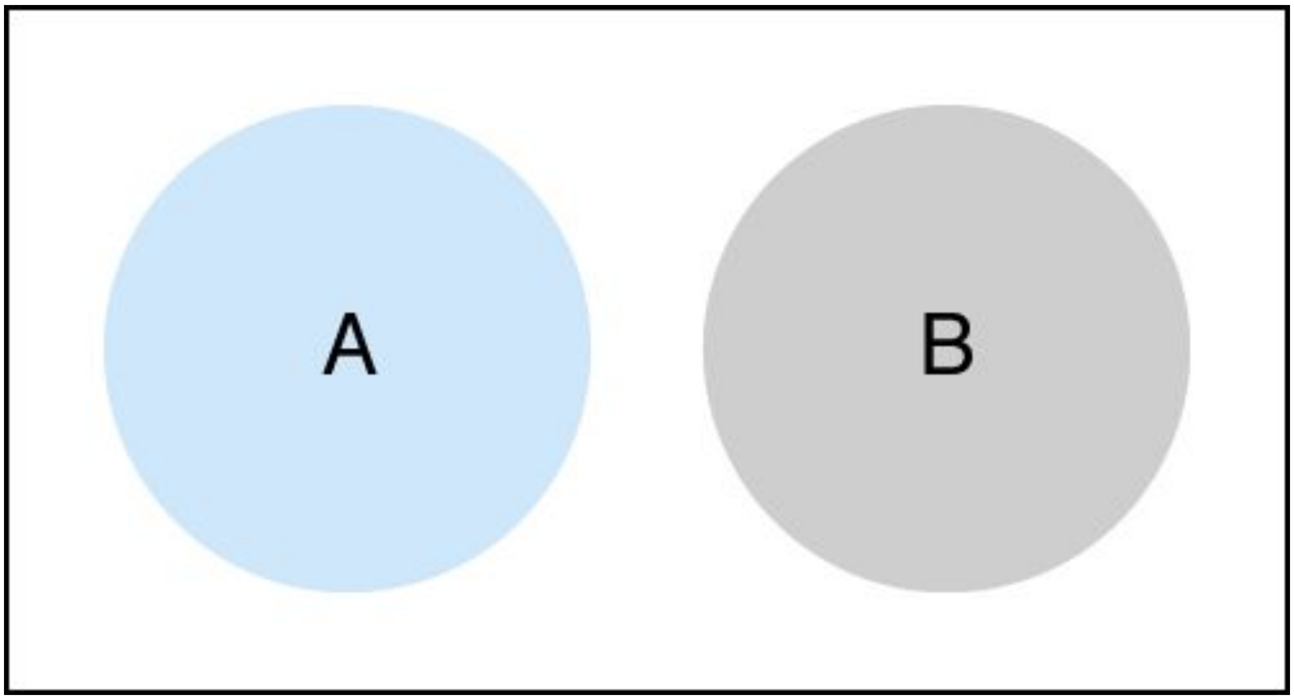


Figure 11.1
Venn Diagram for Mutually Exclusive Events

Fundamentals of Probability
Non-Mutually Exclusive Events & Joint Probability

- Probability that ***non-mutually exclusive events*** A and B or both will occur expressed as:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

- A ***joint probability,* P(AB),** is the probability that two or more events that are not mutually exclusive can occur simultaneously.

Fundamentals of Probability
Non-Mutually Exclusive Events & Joint Probability

M = students taking management science
F = students taking finance

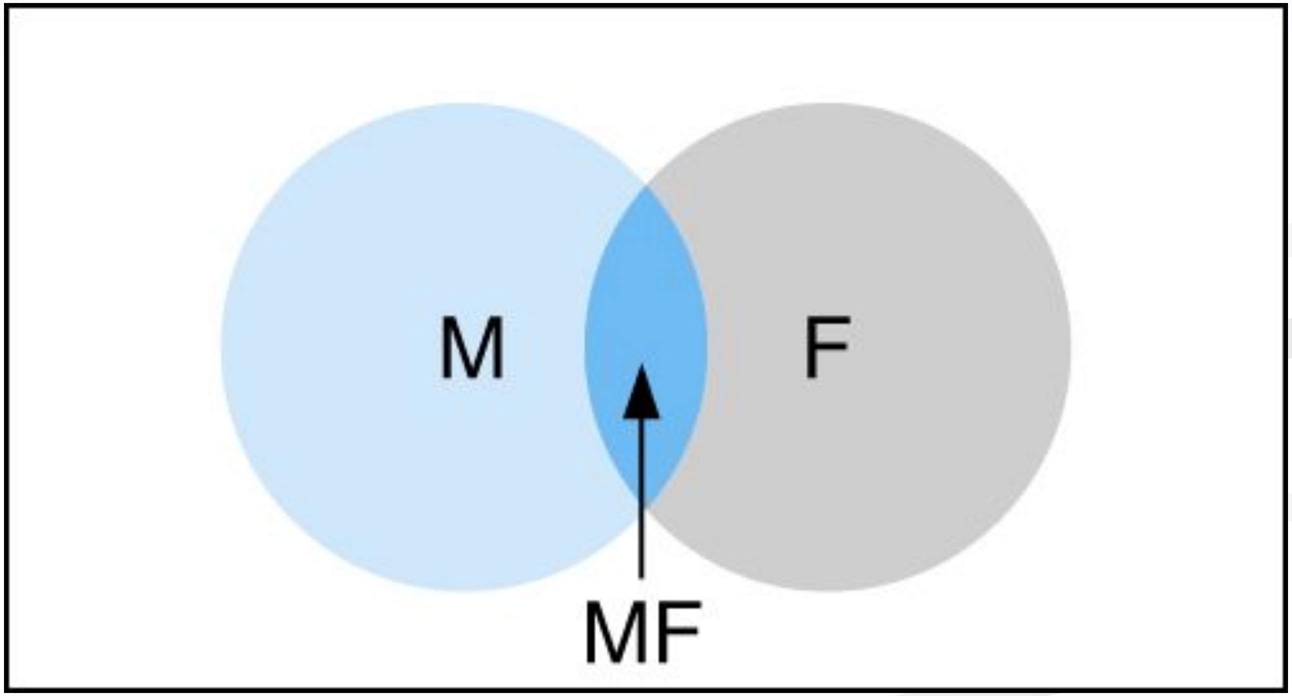


Figure 11.2
Venn diagram for non–mutually exclusive events and the joint event

Fundamentals of Probability
Cumulative Probability Distribution

- Can be developed by adding the probability of an event to the sum of all previously listed probabilities in a probability distribution.

Event Grade	Probability	Cumulative Probability
A	.10	.10
B	.20	.30
C	.50	.80
D	.15	.95
F	.05	1.00
	1.00	

- Probability that a student will get a grade of C or higher:

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) = .10 + .20 + .50 = .80$$

Statistical Independence and Dependence
Independent Events

- A succession of *events that do not affect each other* are *independent events*.
- The *probability of independent events occurring* in a succession is computed by *multiplying the probabilities* of each event.
- A *conditional probability* is the probability that an event will occur given that another event has already occurred, denoted as $P(A|B)$. If events A and B are independent, then:

$$P(AB) = P(A) \cdot P(B) \text{ and } P(A|B) = P(A)$$

Statistical Independence and Dependence
Independent Events – Probability Trees

For coin tossed three consecutive times

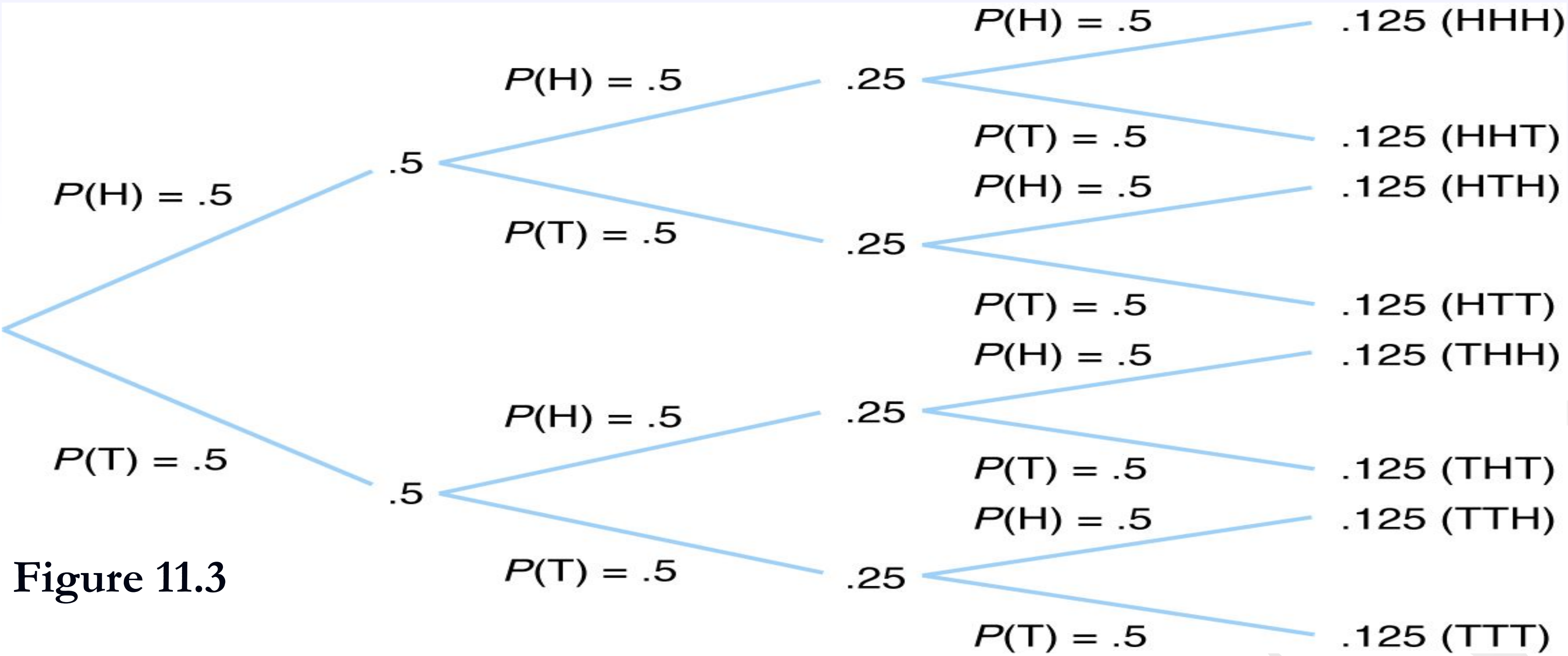


Figure 11.3

Probability of getting head on 1st toss, tail on 2nd, tail on 3rd is:

$$P(HTT) = P(H) \cdot P(T) \cdot P(T) = (.5)(.5)(.5) = .125$$

Statistical Independence and Dependence
Independent Events – Bernoulli Process Definition

Properties of a Bernoulli Process:

- There are *two possible outcomes* for each trial.
- The *probability* of the outcome *remains constant* over time.
- The *outcomes* of the trials are *independent*.
- The *number* of trials is *discrete and integer*.

Statistical Independence and Dependence
Independent Events – Binomial Distribution

- A ***binomial probability distribution function*** is used to determine the probability of a number of successes in *n* trials.
- It is a ***discrete probability distribution*** since the number of successes and trials is discrete.

$$P(r) = \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

where: p = probability of a success

q = 1- p = probability of a failure

n = number of trials

r = number of successes in n trials

Statistical Independence and Dependence Binomial Distribution Example – Tossed Coins

Determine probability of getting exactly two tails in three tosses of a coin.

$$P(2 \text{ tails}) = P(r=2) = \frac{3!}{2!(3-2)!}(.5)^{(2)}(.5)^{(3-2)}$$

$$= \frac{(3 \cdot 2 \cdot 1)}{(2 \cdot 1)(1)}(.25)(.5)$$

$$= \frac{6}{2}(.125)$$

$$P(r=2) = .375$$

Statistical Independence and Dependence Binomial Distribution Example – Quality Control

- Microchip production; sample of four items per batch, 20% of all microchips are defective.
- What is the probability that each batch will contain exactly two defectives?

$$
\begin{aligned}
P(r = 2 \text{ defectives}) &= \frac{4!}{2!(4-2)!}(.2)^2(.8)^2 \\
&= \frac{(4 \cdot 3 \cdot 2 \cdot 1)}{(2 \cdot 1)(1)}(.25)(.5) \\
&= \frac{24}{2}(.0256) \\
&= .1536
\end{aligned}
$$

Statistical Independence and Dependence
Binomial Distribution Example – Quality Control

- Four microchips tested per batch; if two or more found defective, batch is rejected.
- What is probability of rejecting entire batch if batch in fact has 20% defective?

$$P(r \geq 2) = \frac{4!}{2!(4\text{-}2)!}(.2)^2(.8)^2 + \frac{4!}{3!(4-3)!}(.2)^3(.8)^1 + \frac{4!}{4!(4\text{-}4)!}(.2)^4(.8)^0$$
$$= .1536 + .0256 + .0016$$
$$= .1808$$

- Probability of less than two defectives:

$P(r<2) = P(r=0) + P(r=1) = 1.0 - [P(r=2) + P(r=3) + P(r=4)]$

$= 1.0 - .1808 = .8192$

Statistical Independence and Dependence
Dependent Events (1 of 2)

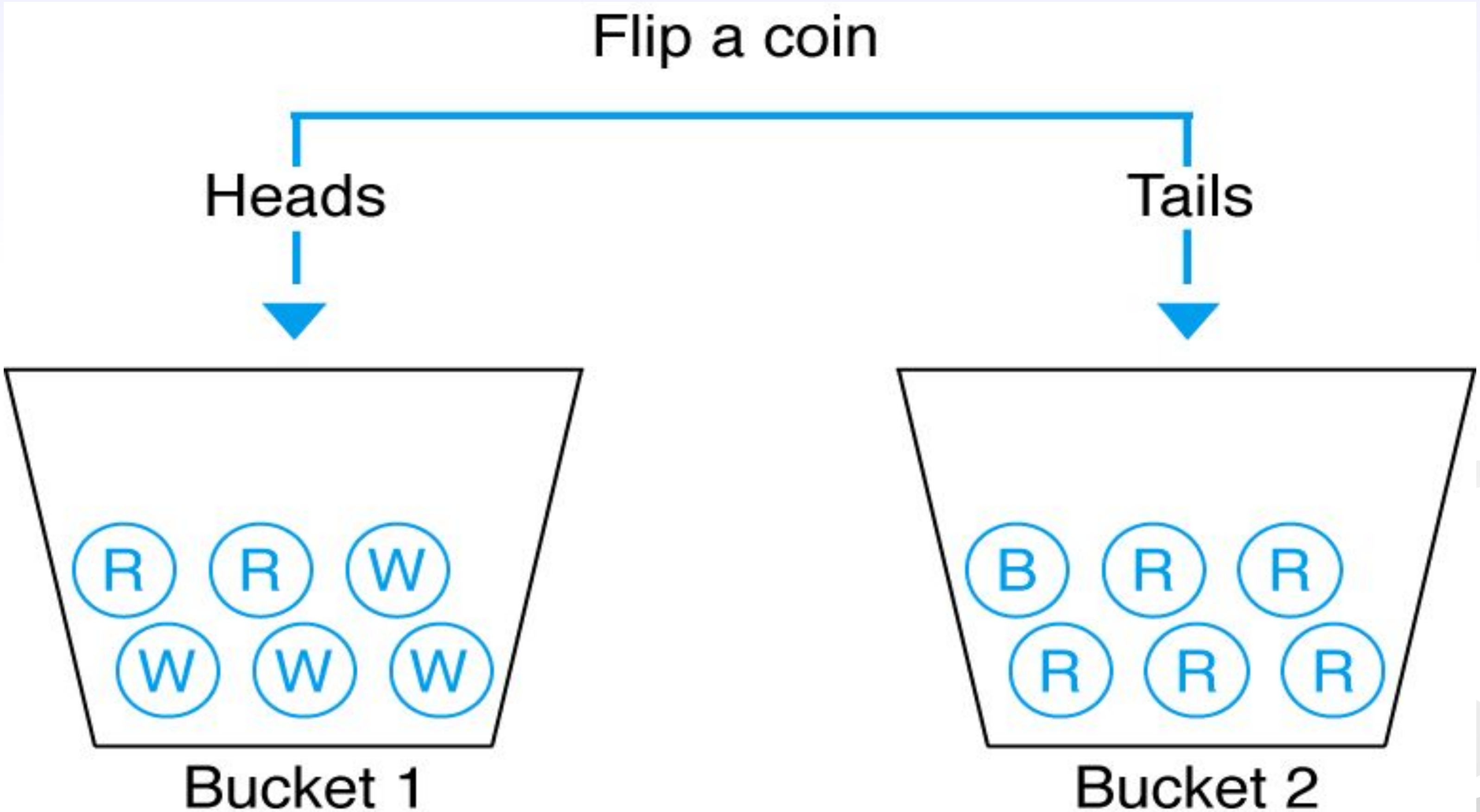


Figure 11.4 Dependent events

Statistical Independence and Dependence
Dependent Events (2 of 2)

- If the *occurrence of one event affects the probability* of the occurrence of another event, the events are *dependent.*

- Coin toss to select bucket, draw for blue ball.

- If tail occurs, 1/6 chance of drawing blue ball from bucket 2; if head results, no possibility of drawing blue ball from bucket 1.

- Probability of event "drawing a blue ball" dependent on event "flipping a coin."

Statistical Independence and Dependence
Dependent Events – Unconditional Probabilities

Unconditional: P(H) = .5; P(T) = .5, must sum to one.

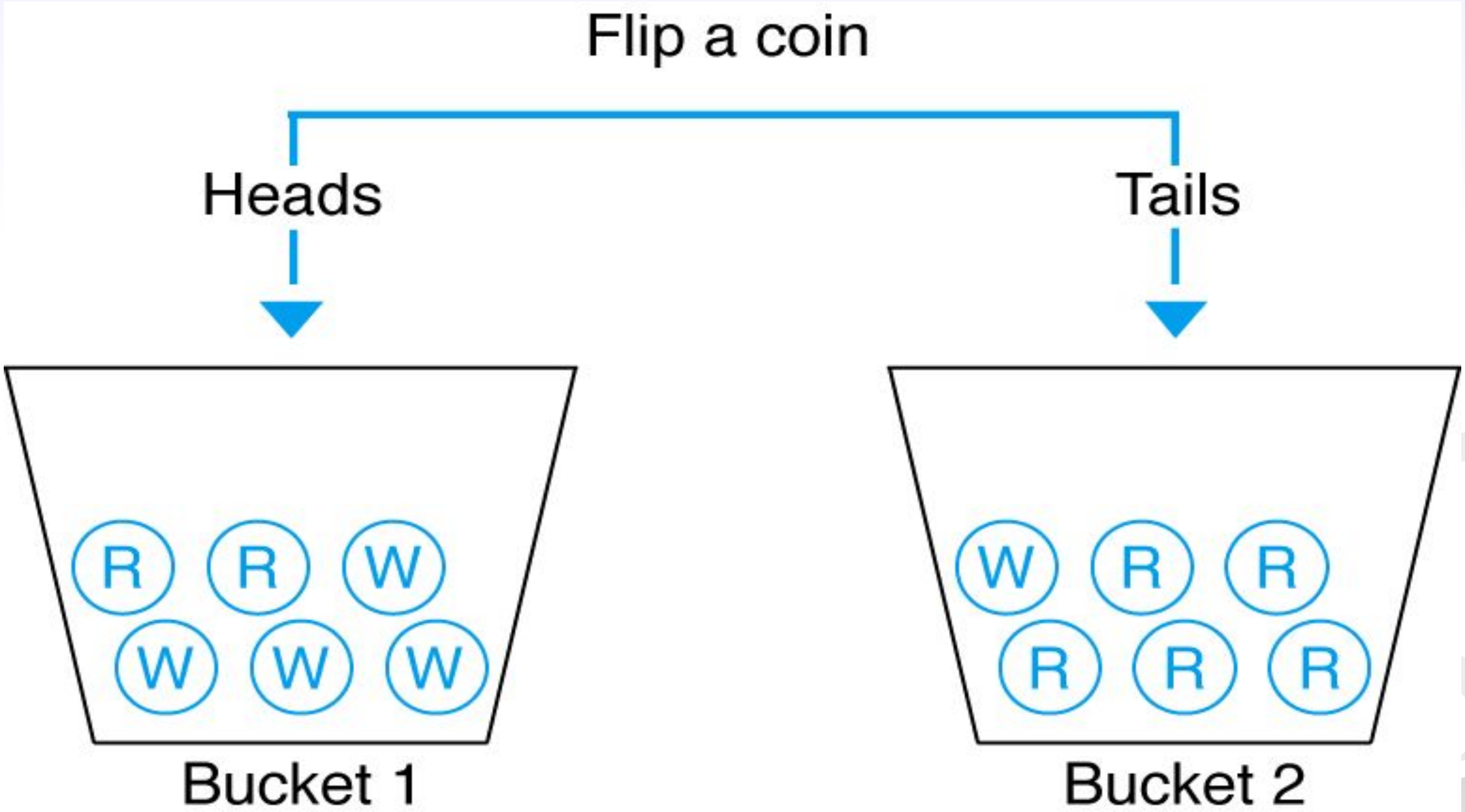


Figure 11.5 Another set of dependent events

Statistical Independence and Dependence
Dependent Events – Conditional Probabilities

Conditional: P(R|H) =.33, P(W|H) = .67, P(R|T) = .83, P(W|T) = .17

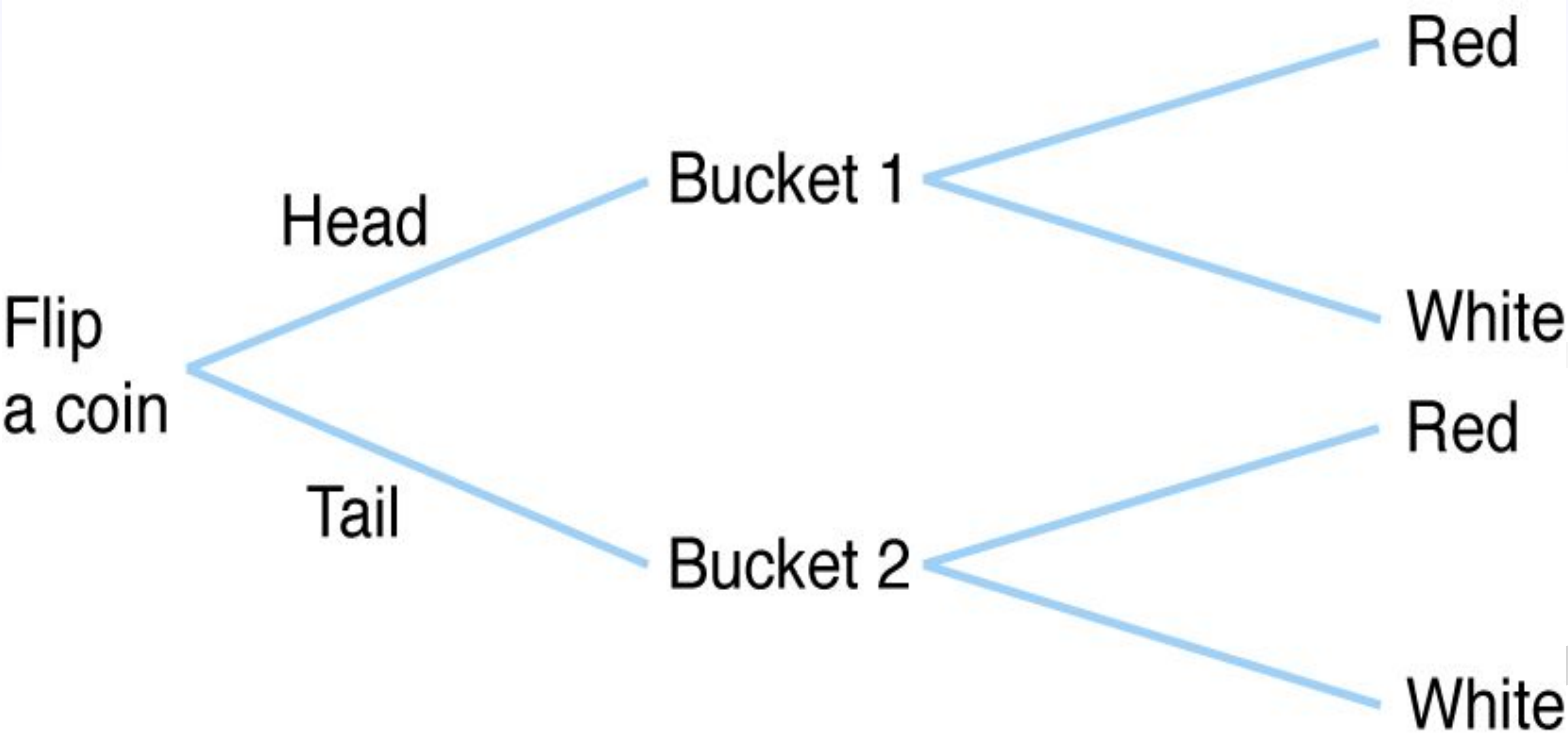


Figure 11.6 Probability tree for dependent events

Statistical Independence and Dependence Math Formulation of Conditional Probabilities

- **Given two dependent events A and B:**

$$P(A|B) = P(AB)/P(B)$$

- **With data from previous example:**

$$P(RH) = P(R|H) \cdot P(H) = (.33)(.5) = .165$$

$$P(WH) = P(W|H) \cdot P(H) = (.67)(.5) = .335$$

$$P(RT) = P(R|T) \cdot P(T) = (.83)(.5) = .415$$

$$P(WT) = P(W|T) \cdot P(T) = (.17)(.5) = .085$$

Statistical Independence and Dependence Summary of Example Problem Probabilities

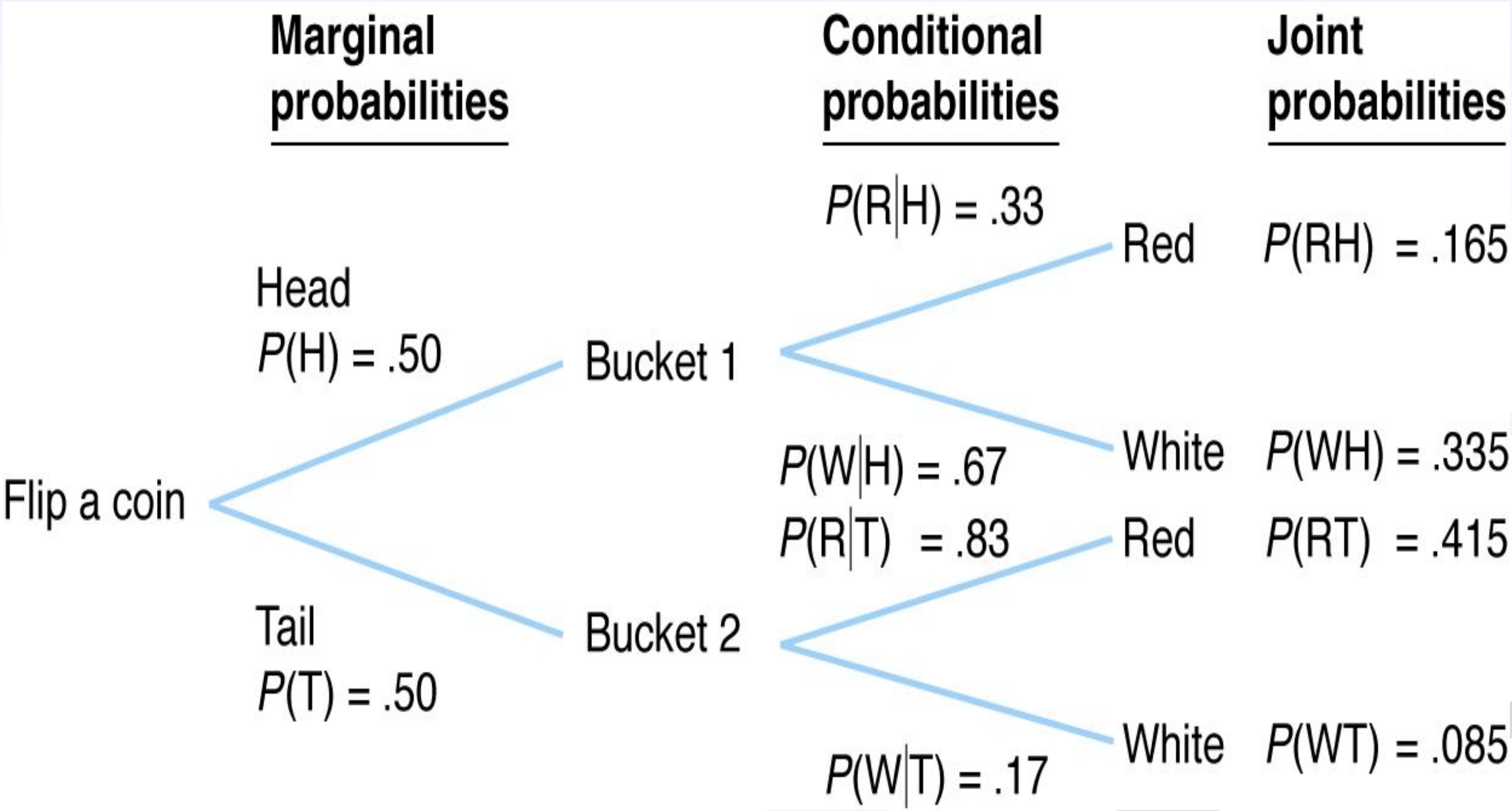


Figure 11.7 Probability tree with marginal, conditional and joint probabilities

Statistical Independence and Dependence Summary of Example Problem Probabilities

Flip a Coin	Draw a Ball: Red	White	Marginal Probabilities
Head	$P(RH) = .165$	$P(WH) = .335$	$P(H) = .50$
Tail	$P(RT) = .415$	$P(WT) = .085$	$P(T) = .50$
Marginal probabilities	$P(R) = .580$	$P(W) = .420$	1.00

Table 11.1 Joint probability table

Statistical Independence and Dependence
Bayesian Analysis

- In Bayesian analysis, *additional information is used to alter the marginal probability* of the occurrence of an event.
- A *posterior probability is the altered marginal probability* of an event based on additional information.
- Bayes' Rule for two events, A and B, and third event, C, conditionally dependent on A and B:

$$P(A|C) = \frac{P(C|A)P(A)}{P(C|A)P(A)+P(C|B)P(B)}$$

Statistical Independence and Dependence Bayesian Analysis – Example (1 of 2)

- Machine setup; if correct there is a 10% chance of a defective part; if incorrect, a 40% chance of a defective part.
- 50% chance setup will be correct or incorrect.
- What is probability that machine setup is incorrect if a sample part is defective?

- Solution: P(C) = .50, P(IC) = .50, P(D|C) = .10, P(D|IC) = .40

  where C = correct, IC = incorrect, D = defective

Statistical Independence and Dependence Bayesian Analysis – Example (2 of 2)

Posterior probabilities:

$$P(IC|D) = \frac{P(D|IC)P(IC)}{P(D|IC)P(IC)+P(D|C)P(C)}$$

$$= \frac{(.40)(.50)}{(.40)(.50)+(.10)(.50)}$$

$$= .80$$

Expected Value
Random Variables

- When the values of variables occur in *no particular order* or sequence, the variables are referred to as *random variables.*
- Random variables are represented symbolically by a letter x, y, z, etc.
- Although *exact values of random variables are not known* prior to events, it is possible to assign a probability to the occurrence of possible values.

Expected Value Example (1 of 4)

- Machines break down 0, 1, 2, 3, or 4 times per month.
- Relative frequency of breakdowns , or a *probability distribution*:

Random Variable x (Number of Breakdowns)	P(x)
0	.10
1	.20
2	.30
3	.25
4	.15
	1.00

Expected Value Example (2 of 4)

- The *expected value* of a random variable is computed by *multiplying each possible value of the variable by its probability and summing* these products.
- The *expected value is the weighted average, or mean*, of the probability distribution of the random variable.
- Expected value of number of breakdowns per month:

$$E(x) = (0)(.10) + (1)(.20) + (2)(.30) + (3)(.25) + (4)(.15)$$

$$= 0 + .20 + .60 + .75 + .60$$

$$= 2.15 \text{ breakdowns}$$

Expected Value Example (3 of 4)

- *Variance is a measure of the dispersion* of a random variable's values about the mean.
- Variance is computed as follows:
  1. Square the difference between each value and the expected value.
  2. Multiply the resulting amounts by the probability of each value.
  3. Sum the values compiled in step 2.

General formula:

$$\sigma^2 = \sum_{i=1}^{n} [x_i - E(x)]^2 P(x_i)$$

Expected Value Example (4 of 4)

- Standard deviation is computed by taking the square root of the variance.
- For example data $[E(x) = 2.15]$:

x_i	$P(x_i)$	$x_i - E(x)$	$[x_i - E(x)]^2$	$[x_i - E(x)]^2 \bullet P(x_i)$
0	.10	-2.15	4.62	.462
1	.20	-1.15	1.32	.264
2	.30	-0.15	0.02	.006
3	.25	0.85	0.72	.180
4	.15	1.85	3.42	.513
	1.00			1.425

$\sigma^2 = 1.425$ (breakdowns per month)^2

standard deviation $= \sigma =$ sqrt(1.425)

$= 1.19$ breakdowns per month

The Normal Distribution
Continuous Random Variables

- A *continuous random variable* can take on an infinite number of values within some interval.

- Continuous random variables have *values that are not specifically countable* and are often fractional.

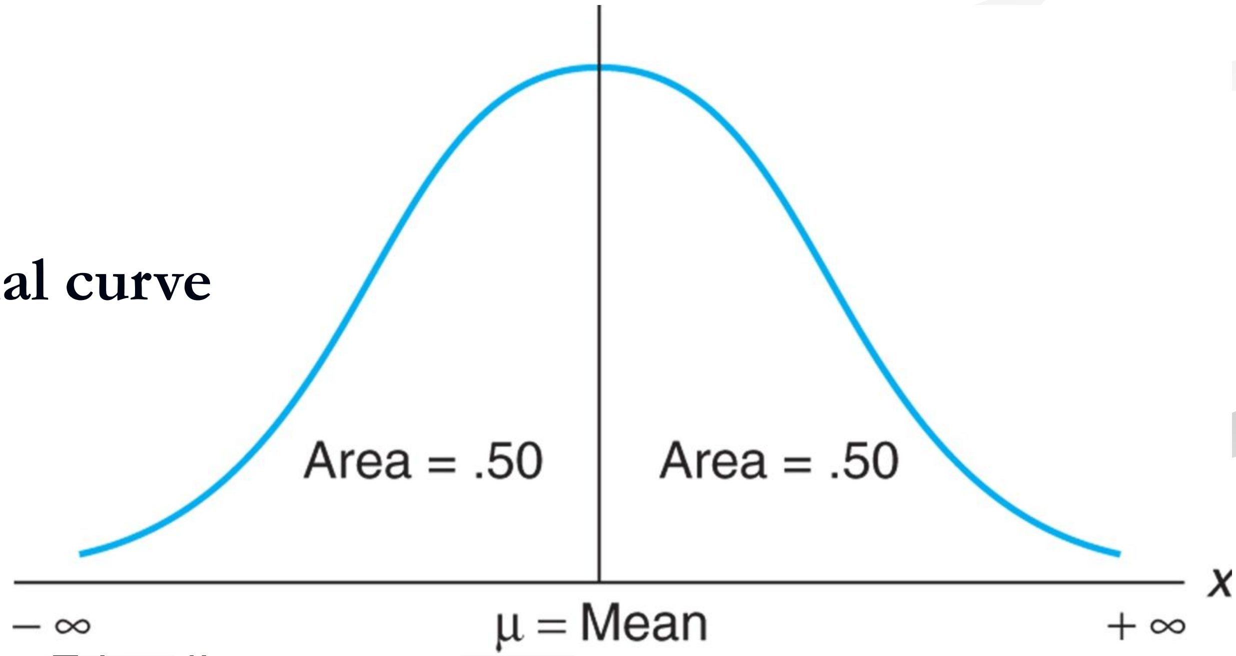
- *Cannot assign a unique probability* to each value of a continuous random variable.

- In a *continuous probability distribution* the probability refers to a value of the random variable being within some *range.*

The Normal Distribution Definition

- The ***normal distribution*** is a continuous probability distribution that is symmetrical on both sides of the mean.
- The center of a normal distribution is its ***mean μ***.
- The ***area under the normal curve*** represents probability, and the total area under the curve ***sums to one***.

Figure 11.8 The normal curve



The Normal Distribution Example (1 of 5)

- **Mean weekly carpet sales of 4,200 yards, with a standard deviation of 1,400 yards.**
- **What is the probability of sales exceeding 6,000 yards?**
- **$\mu = 4{,}200$ yd; $\sigma = 1{,}400$ yd; probability that number of yards of carpet will be equal to or greater than 6,000 expressed as: $P(x \geq 6{,}000)$.**

The Normal Distribution Example (2 of 5)

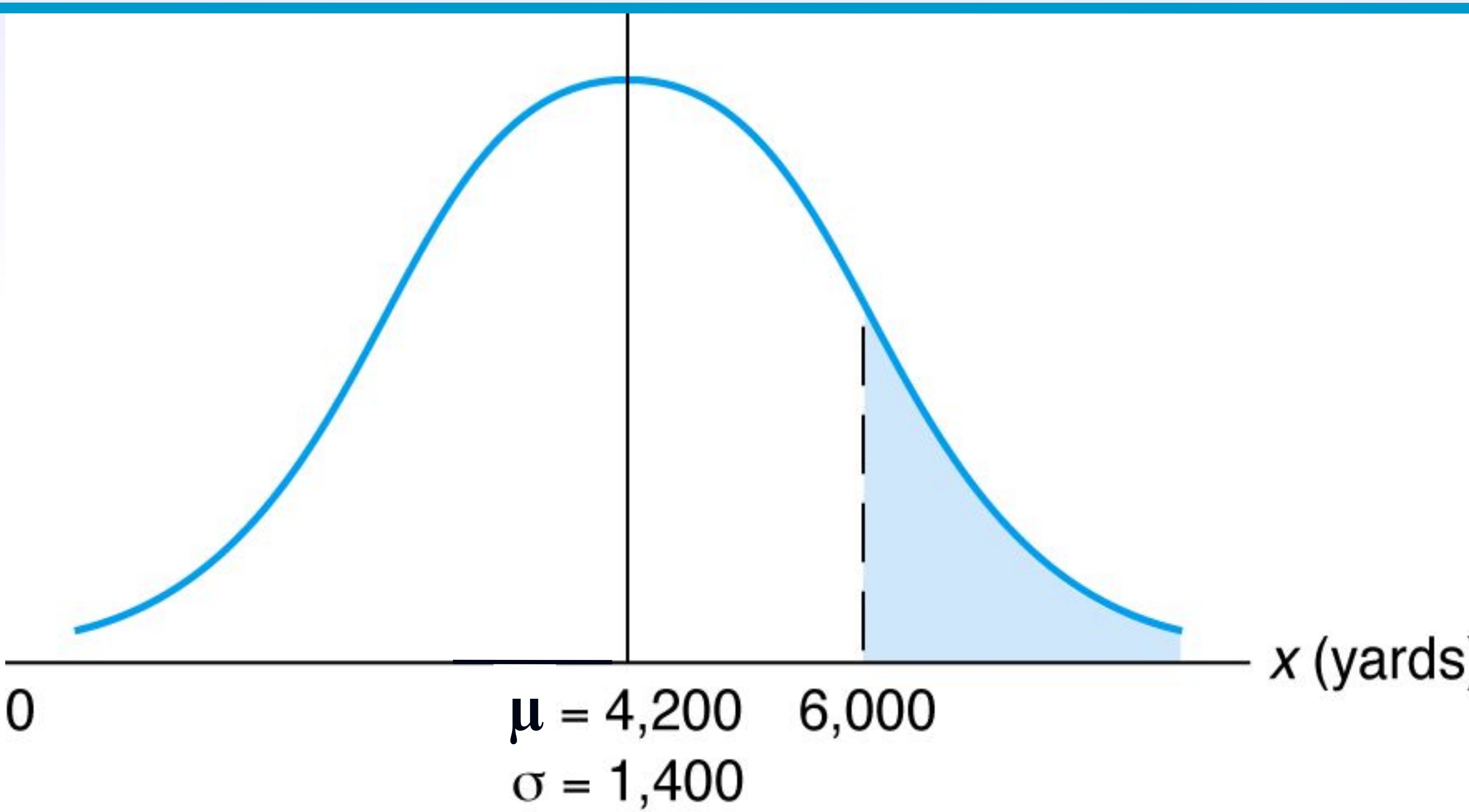


Figure 11.9 The normal distribution for carpet demand

The Normal Distribution
Standard Normal Curve (1 of 2)

- The area or probability under a normal curve is measured by determining the number of standard deviations the value of a random variable x is from the mean.
- Number of standard deviations a value is from the mean designated as Z.

$$Z = \frac{X - \mu}{\sigma}$$

The Normal Distribution
Standard Normal Curve (2 of 2)

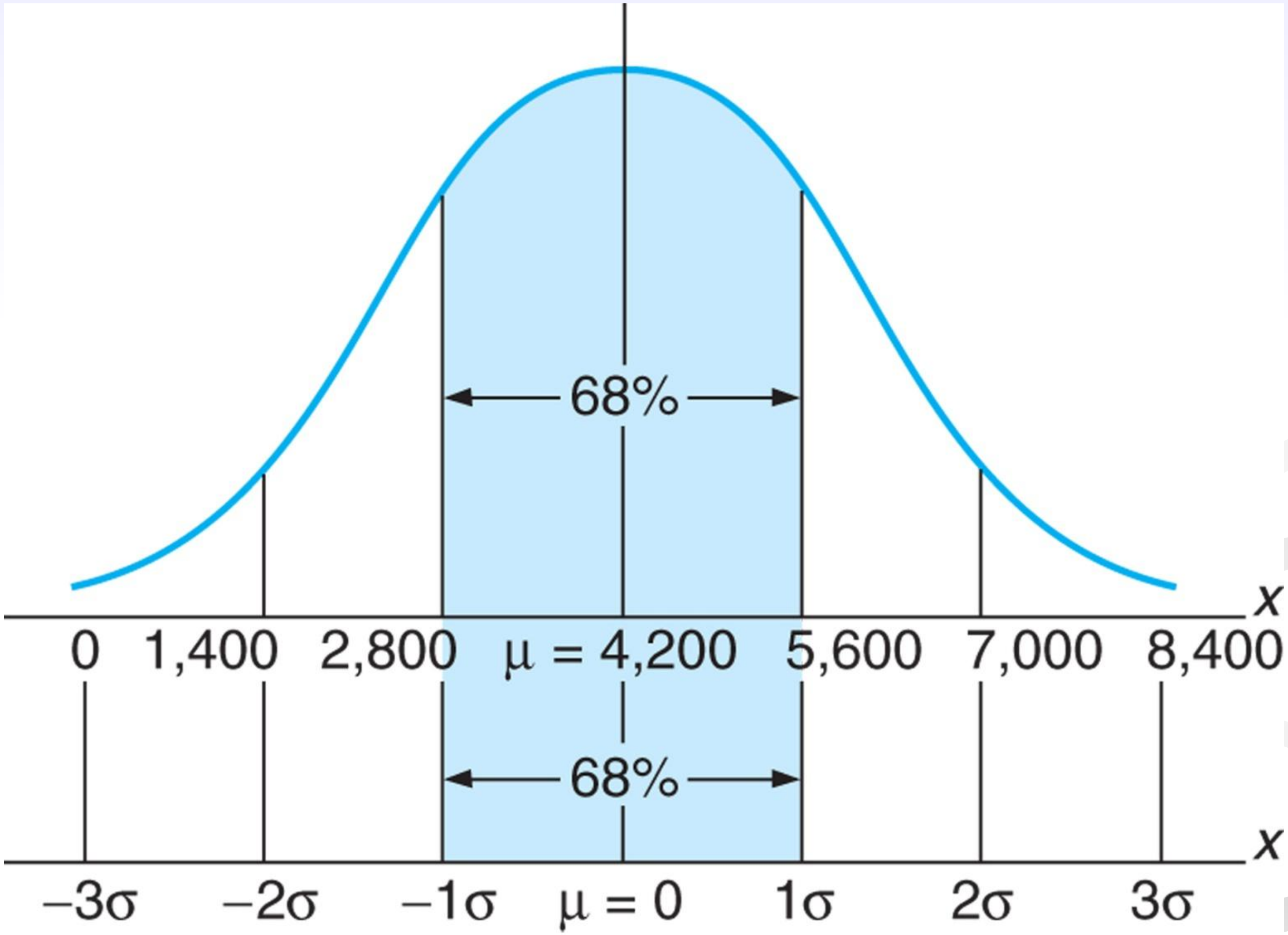


Figure 11.10 The standard normal distribution

The Normal Distribution Example (3 of 5)

$Z = (x - \mu)/\sigma = (6{,}000 - 4{,}200)/1{,}400$

$= 1.29$ standard deviations

$P(x \geq 6{,}000) = .5000 - .4015 = .0985$

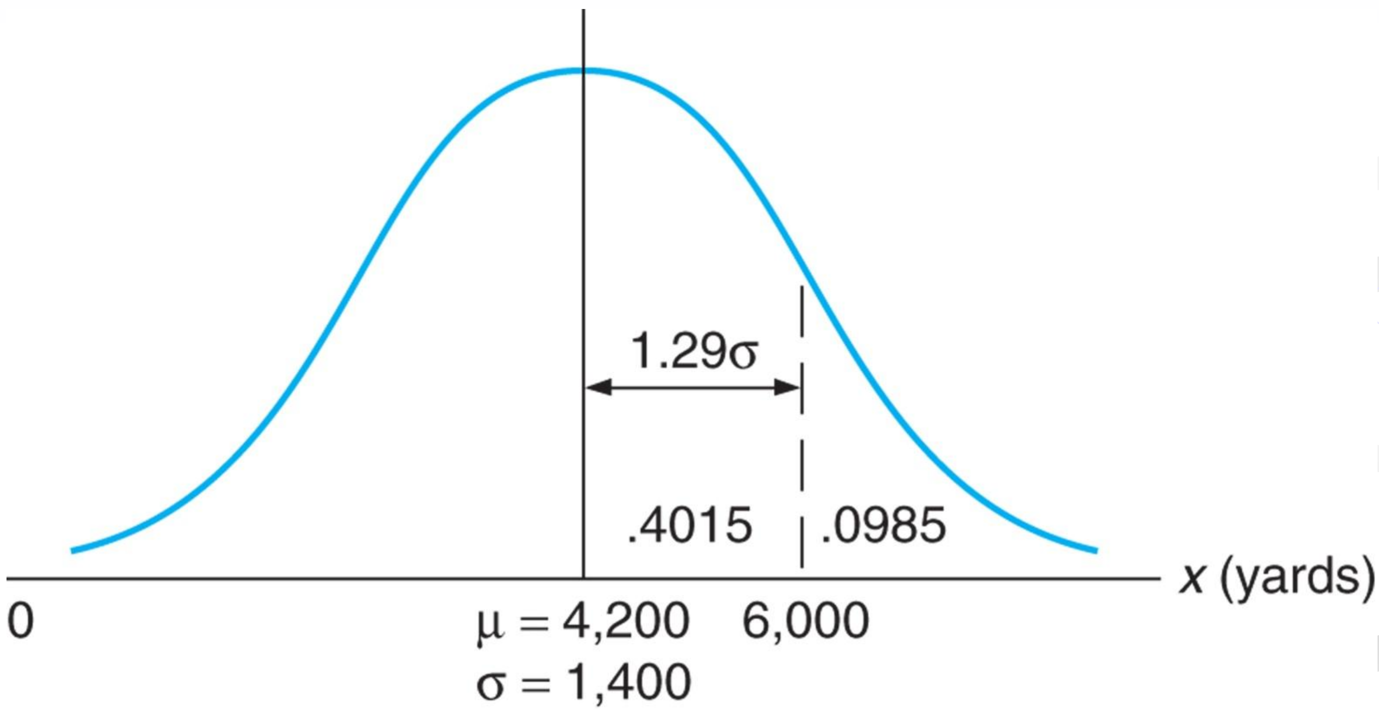


Figure 11.11 Determination of the Z value

The Normal Distribution Example (4 of 5)

Determine the probability that demand will be 5,000 yards or less.

$Z = (x - \mu)/\sigma = (5{,}000 - 4{,}200)/1{,}400 = .57$ standard deviations

$P(x \leq 5{,}000) = .5000 + .2157 = .7157$

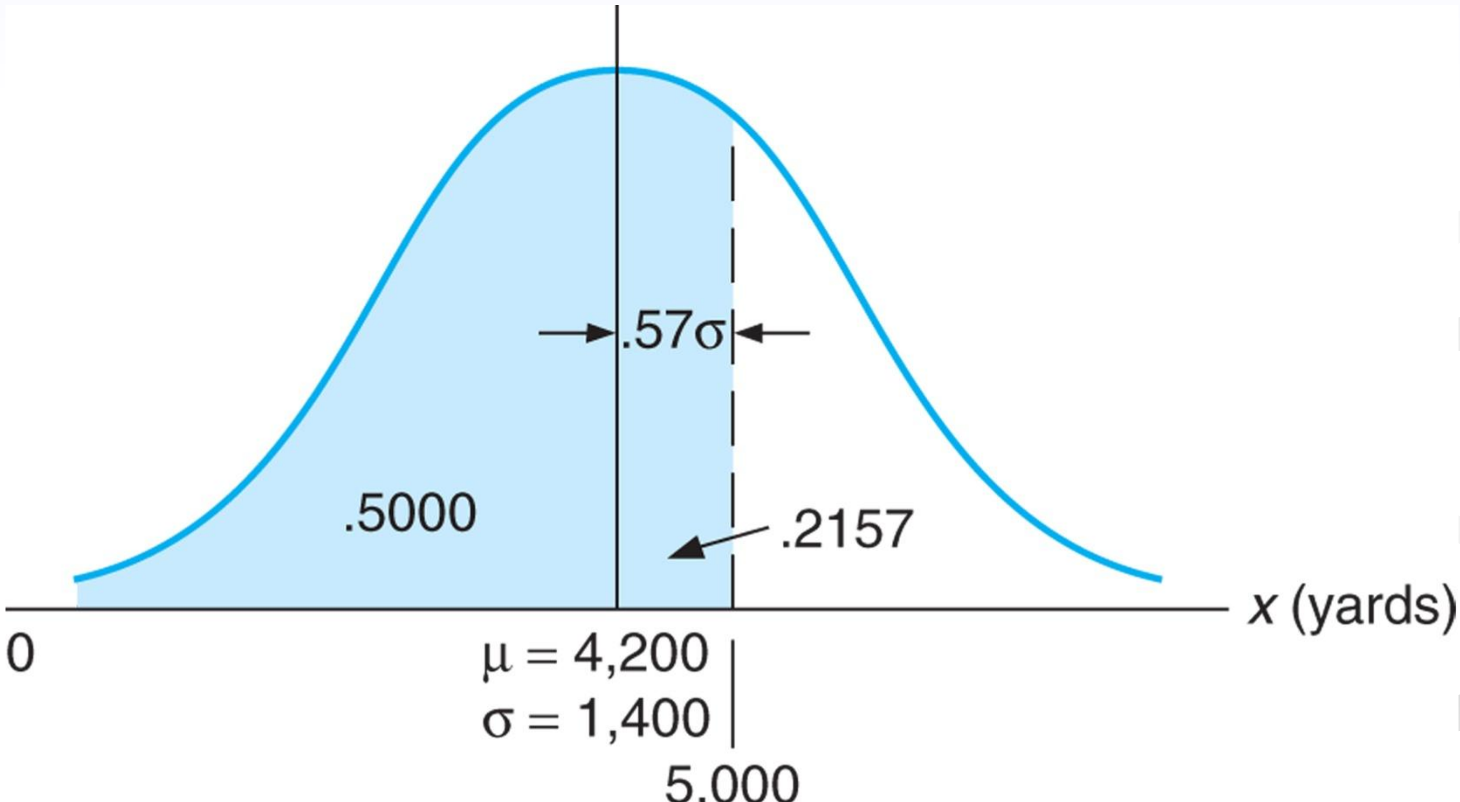


Figure 11.12 Normal distribution for P(x ≤ 5,000 yards)

The Normal Distribution Example (5 of 5)

Determine the probability that demand will be between 3,000 yards and 5,000 yards.

$Z = (3{,}000 - 4{,}200)/1{,}400 = -1{,}200/1{,}400 = -.86$

$P(3{,}000 \leq x \leq 5{,}000) = .2157 + .3051 = .5208$

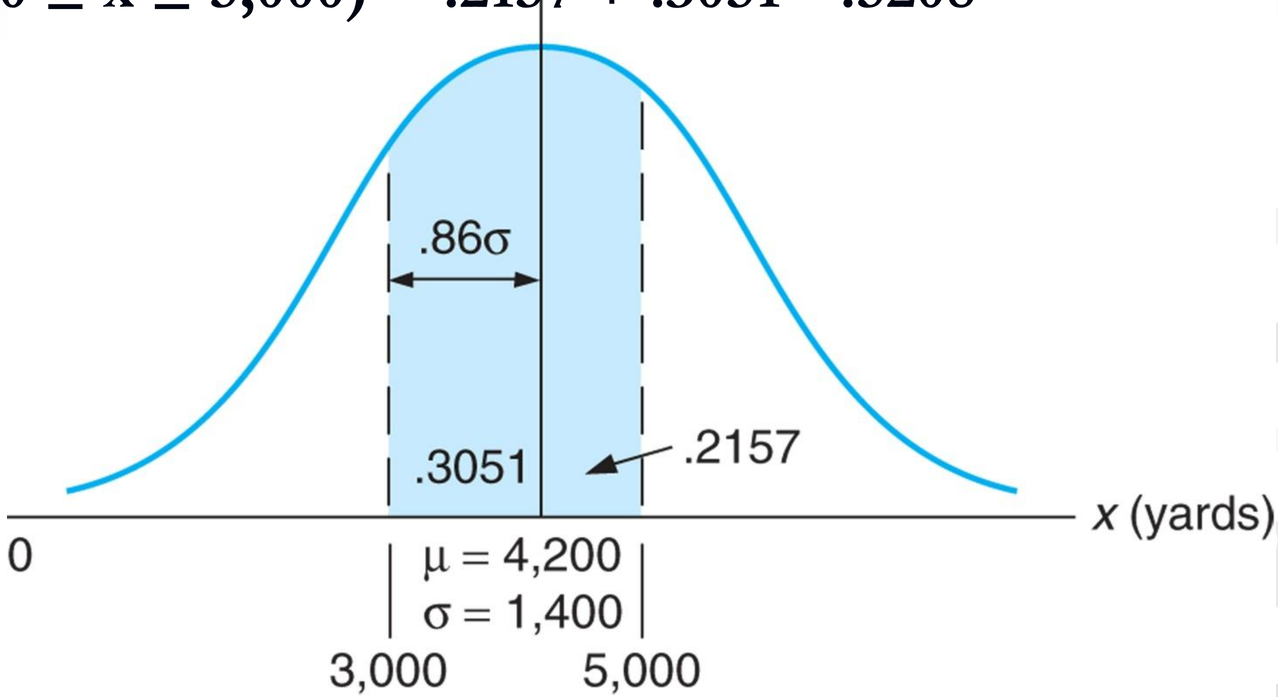


Figure 11.13 Normal distribution with P(3000 yards ≤ x ≤ 5000 yards)

The Normal Distribution
Sample Mean and Variance

- The *population* mean and variance are for the *entire set* of data being analyzed.

- The *sample* mean and variance are derived *from a subset* of the population data and are used to make inferences about the population.

The Normal Distribution
Computing the Sample Mean and Variance

Sample mean $$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Sample variance $$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$$

Sample variance shortcut form $$s^2 = \frac{\sum_{i=1}^{n} x_i^{\ 2} - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n-1}$$

The Normal Distribution Example Problem Revisited

Sample mean = 42,000/10 = 4,200 yd

Sample variance = [(190,060,000) - (1,764,000,000/10)]/9

= 1,517,777

Sample std. dev. = sqrt(1,517,777)

= 1,232 yd

Week i	Demand x_i
1	2,900
2	5,400
3	3,100
4	4,700
5	3,800
6	4,300
7	6,800
8	2,900
9	3,600
10	4,500
	42,000

The Normal Distribution
Chi-Square Test for Normality (1 of 2)

- It can *never be simply assumed that data are normally distributed.*
- The *chi-square test* is used to determine if a set of *data fit a particular distribution.*
- The *chi-square test* compares an observed frequency distribution with a theoretical frequency distribution (testing the *goodness-of-fit*).

The Normal Distribution
Chi-Square Test for Normality (2 of 2)

- In the test, the *actual number of frequencies* in each range of frequency distribution is *compared to the theoretical frequencies* that should occur in each range if the data follow a particular distribution.
- A chi-square statistic is then calculated and compared to a number, called a *critical value,* from a chi-square table.
- If the *test statistic is greater* than the critical value, the *distribution does not follow* the distribution being tested; if it is less, the distribution fits.
- The chi-square test is a form of *hypothesis testing*.

The Normal Distribution Example of Chi-Square Test (1 of 6)

Armor Carpet Store example - assume sample mean = 4,200 yards, and sample standard deviation =1,232 yards.

Range, Weekly Demand (yds)	Frequency (weeks)
0 – 1,000	2
1,000 – 2,000	5
2,000 – 3,000	22
3,000 – 4,000	50
4,000 – 5,000	62
5,000 – 6,000	40
6,000 – 7,000	15
7,000 – 8,000	3
8,000 +	1
	200

The Normal Distribution
Example of Chi-Square Test (2 of 6)

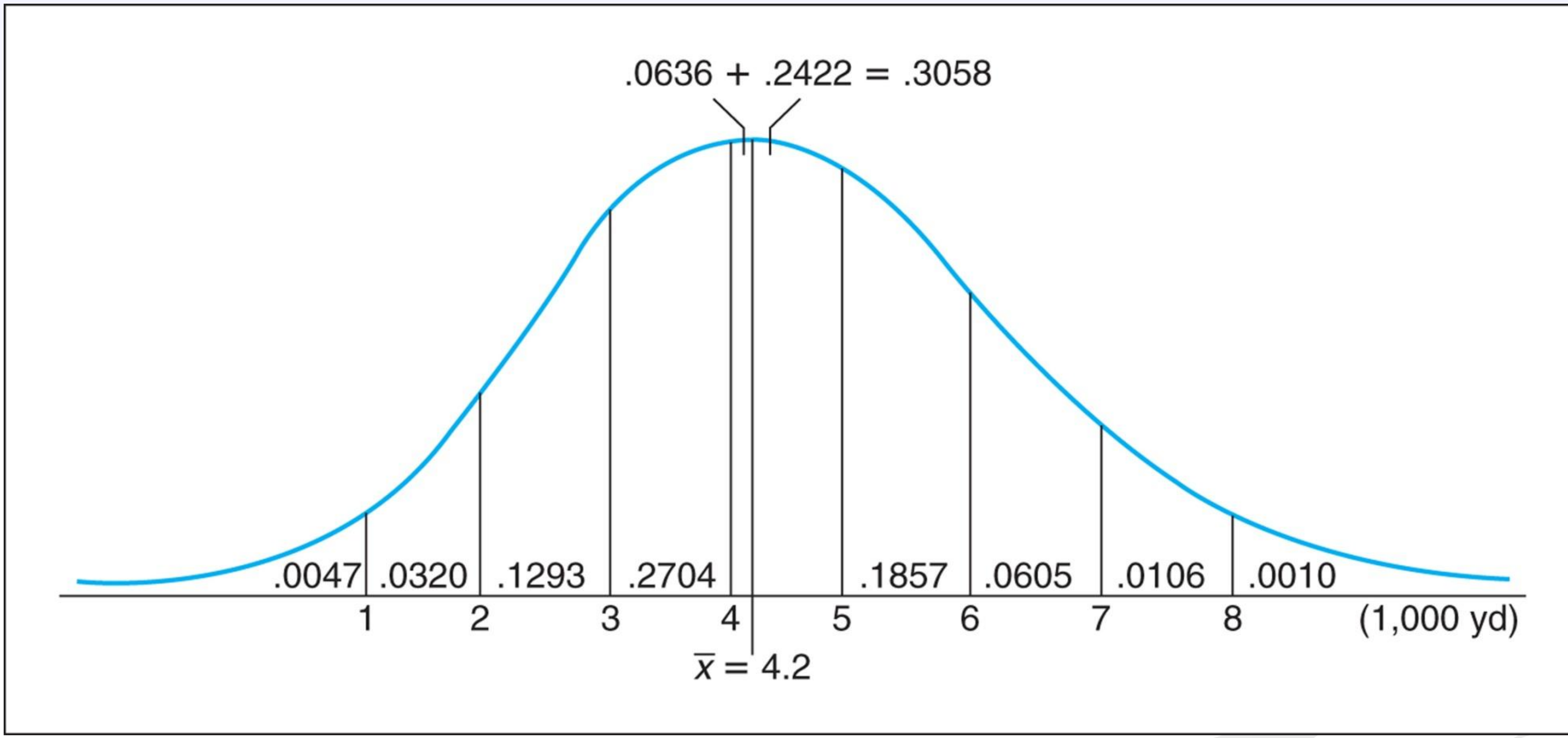


Figure 11.14 The theoretical normal distribution

The Normal Distribution
Example of Chi-Square Test (3 of 6)

Range	Z	Area: $x \rightarrow \bar{x}$	Range Area	Normal Frequency (n = 200)
0–1,000	—	.5000	.0047	0.94
1,000–2,000	−2.60	.4953	.0320	6.40
2,000–3,000	−1.79	.4633	.1293	25.86
3,000–4,000	−.97	.3340	.2704	54.08
4,000–5,000	{ −.16 .65	.0636 .2422 }	.3058	61.16
5,000–6,000	1.46	.4279	.1857	37.14
6,000–7,000	2.27	.4884	.0605	12.10
7,000–8,000	3.08	.4990	.0106	2.12
8,000+	—	.5000	.0010	0.20

Table 11.2 The determination of the theoretical range frequencies

Example of Chi-Square Test (4 of 6)

Comparing theoretical frequencies with actual frequencies:

$$\chi^2_{k-p-1} = \sum_k \frac{\left(f_0 - f_t\right)^2}{f_t}$$

where: f_o = observed frequency

f_t = theoretical frequency

k = the number of classes,

p = the number of estimated parameters

k-p-1 = degrees of freedom.

The Normal Distribution
Example of Chi-Square Test (5 of 6)

Range, Weekly Demand	Observed Frequency f_o	Theoretical Frequency f_t	$(f_o - f_t)^2$	$(f_o - f_t)^2/f_t$
0–2,000	7	7.34	.12	.016
2,000–3,000	22	25.86	14.90	.576
3,000–4,000	50	54.08	16.64	.308
4,000–5,000	62	61.16	.71	.012
5,000–6,000	40	37.14	8.18	.220
6,000+	19	14.42	21.00	1.456
				2.588

Table 11.3 Computation of χ^2 test statistic

The Normal Distribution
Example of Chi-Square Test (6 of 6)

$\chi^2_{k-p-1} = \Sigma(f_o - f_t)^2/10 = 2.588$

k - p -1 = 6 - 2 – 1 = 3 degrees of freedom,

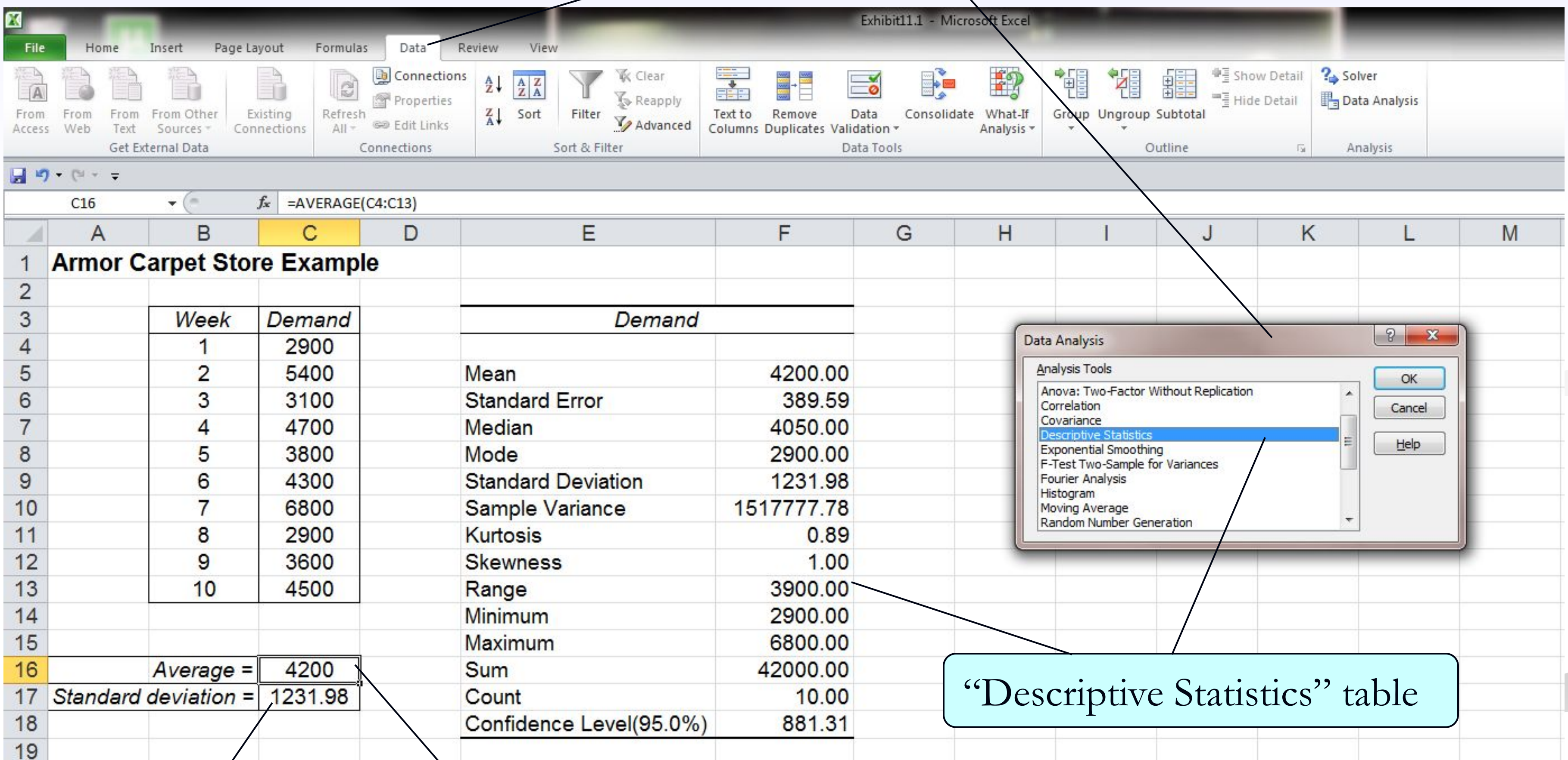
with level of significance (deg of confidence) of .05 ($\alpha = .05$).

from Table A.2, $\chi^2_{.05,3} = 7.815$;

because 7.815 > 2.588, we accept the hypothesis that the distribution is normal.

Statistical Analysis with Excel (1 of 2)

Click on "Data" tab on toolbar; then on "Data Analysis"; then select "Descriptive Statistics"



Armor Carpet Store Example

Week	Demand
1	2900
2	5400
3	3100
4	4700
5	3800
6	4300
7	6800
8	2900
9	3600
10	4500
Average =	4200
Standard deviation =	1231.98

Demand	
Mean	4200.00
Standard Error	389.59
Median	4050.00
Mode	2900.00
Standard Deviation	1231.98
Sample Variance	1517777.78
Kurtosis	0.89
Skewness	1.00
Range	3900.00
Minimum	2900.00
Maximum	6800.00
Sum	42000.00
Count	10.00
Confidence Level(95.0%)	881.31

"Descriptive Statistics" table

=AVERAGE(C4:C13)

=STDEV(C4:C13)

Exhibit 11.1

Statistical Analysis with Excel (2 of 2)

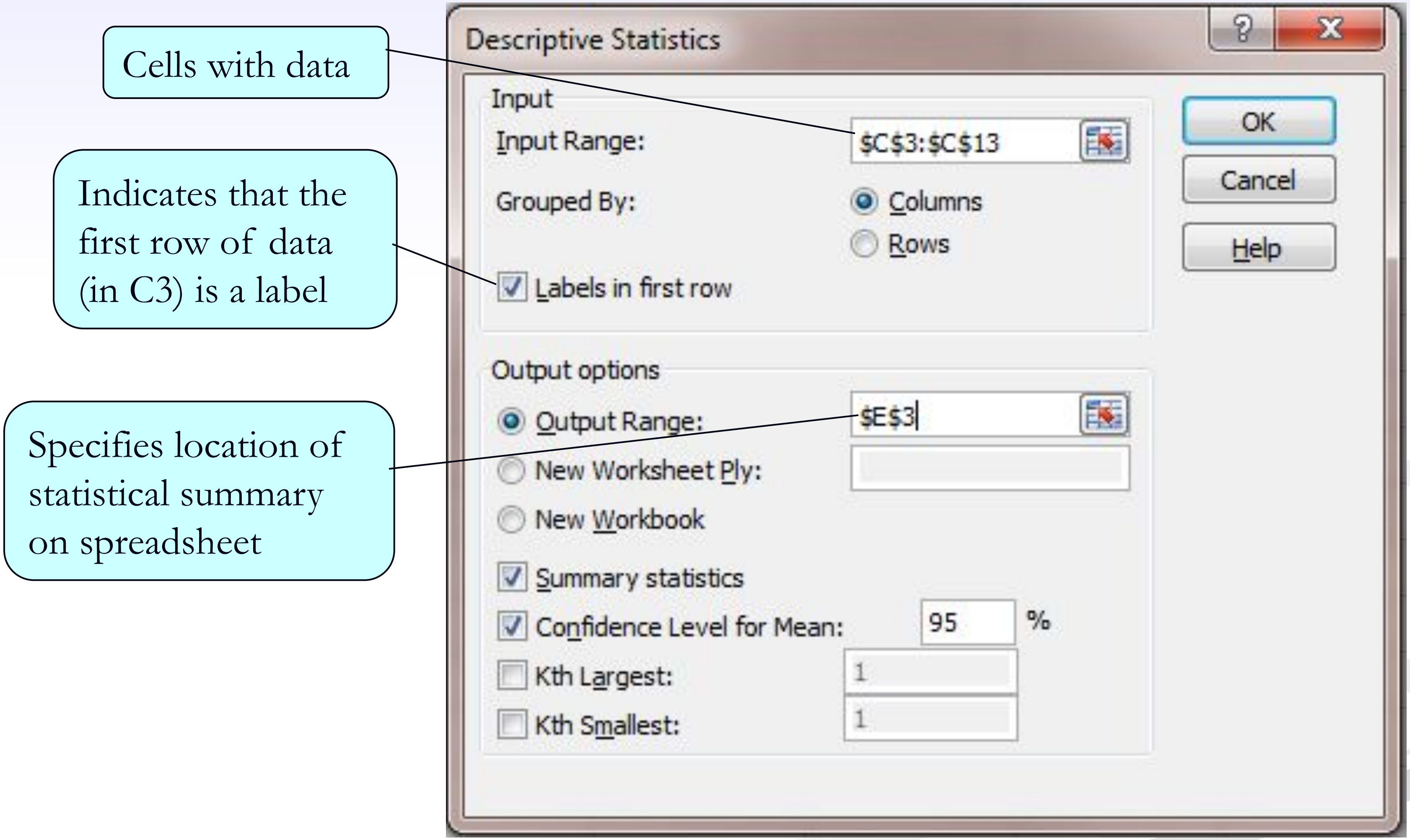


Exhibit 11.2

Example Problem Solution Data

Radcliff Chemical Company and Arsenal.

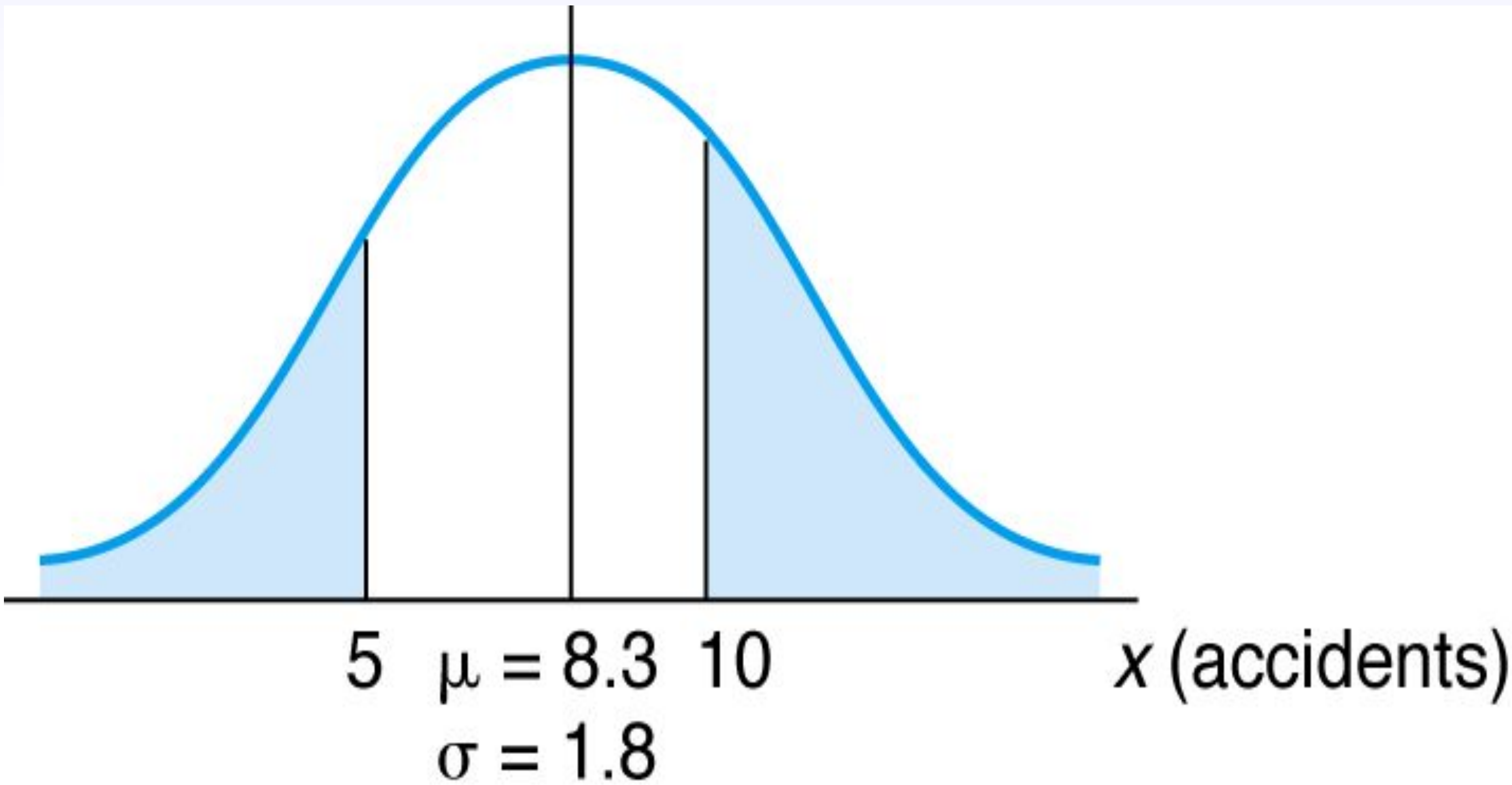
Annual number of accidents is normally distributed with mean of 8.3 and standard deviation of 1.8 accidents.

1. What is the probability that the company will have fewer than five accidents next year? More than ten?

2. The government will fine the company $200,000 if the number of accidents exceeds 12 in a one-year period. What average annual fine can the company expect?

Example Problem Solution
Solution (1 of 3)

Set up the normal distribution.



Example Problem Solution
Solution (2 of 3)

Solve Part 1: P(x ≤ 5 accidents) and P(x ≥ 10 accidents).

$Z = (x - \mu)/\sigma = (5 - 8.3)/1.8 = -1.83$.

From Table A.1, Z = -1.83 corresponds to probability of .4664, and $P(x \leq 5) = .5000 - .4664 = .0336$

$Z = (10 - 8.3)/1.8 = .94$.

From Table A.1, Z = .94 corresponds to probability of .3264 and $P(x \geq 10) = .5000 - .3264 = .1736$

Example Problem Solution
Solution (3 of 3)

Solve Part 2:

P(x ≥ 12 accidents)

Z = 2.06, corresponding to probability of .4803.

P(x ≥ 12) = .5000 - .4803 = .0197, expected annual fine

= $200,000(.0197) = $3,940