Лекция 11
Негармонические волны в равновесных средах с дисперсией

1. Общее решение начальной задачи
   а. Основные формулы для пространственно гармонических волн
   б. Построение общего решения в случае негармонических волн
   в. Пространственно временная функция точечного источника
   г. Задача об эволюции негармонического начального возмущения

2. Квазигармоническое приближение. Групповая скорость
   а. Спектральная плотность начального возмущения
   б. Случай слабой дисперсии – групповая скорость
   в. Скорость переноса энергии волны
   г. Групповая скорость и метод медленных амплитуд
   д. Групповые скорости конкретных волн
   е. Поток энергии волн

3. Расплывание импульсов в равновесных средах с дисперсией
   а. Постановка задачи и общие соображения
   б. Компьютерное моделирование
   в. Метод стационарной фазы

$$A(t,z) = A(t,k)\exp(ikz) \qquad D(\hat{\omega},k)A(t,k) = F(t,k)$$

$$A(t,k) = \frac{1}{2\pi}\int_{C(\omega)} \frac{P_{n-1}(\omega,k)}{D(\omega,k)}\exp(-i\omega t)d\omega \qquad A(t,k) = \sum_{m=1}^{n}\left[A_m(k)\exp(-i\omega_m(k)t)\right]$$

$$A(t,k) = \int_0^t G(\tau,k)F(t-\tau,k)d\tau \qquad G(t,k) = \frac{1}{2\pi}\int_{C(\omega)} \frac{1}{D(\omega,k)}\exp(-i\omega t)d\omega$$

$$D(\hat{\omega},\hat{k})A(t,z) = F(t,z)$$

$$A(t,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(t,k)\exp(ikz)dk \quad \rightleftarrows \quad A(t,k) = \int_{-\infty}^{+\infty} A(t,z)\exp(-ikz)dz$$

$$A(t,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} dk \int_{C(\omega)} \frac{P_{n-1}(\omega,k)}{D(\omega,k)}\exp(-i\omega t + ikz)d\omega$$

$$A(t,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{m=1}^{n}\left[A_m(k)\exp(-i\omega_m(k)t + ikz)\right]dk$$

$$A(t,z) = \int_0^t d\tau \int_{-\infty}^{+\infty} dx\, G(\tau,x)F(t-\tau,z-x) \qquad D(\hat{\omega},\hat{k})G(t,z) = \delta(t)\delta(z)$$

$$G(t,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(t,k)\exp(ikz)dk = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} dk \int_{C(\omega)} \frac{1}{D(\omega,k)}\exp(-i\omega t + ikz)d\omega$$

$$A(t,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}A(k)\exp\left(-i\omega(k)t+ikz\right)dk$$

$$A(0,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}A(k)\exp(ikz)dk\equiv A_0(z)\qquad \Delta z\cdot\Delta k\sim 2\pi\qquad A(k)=\int_{-\infty}^{+\infty}A_0(z)\exp(-ikz)dz\equiv A_0(k)$$

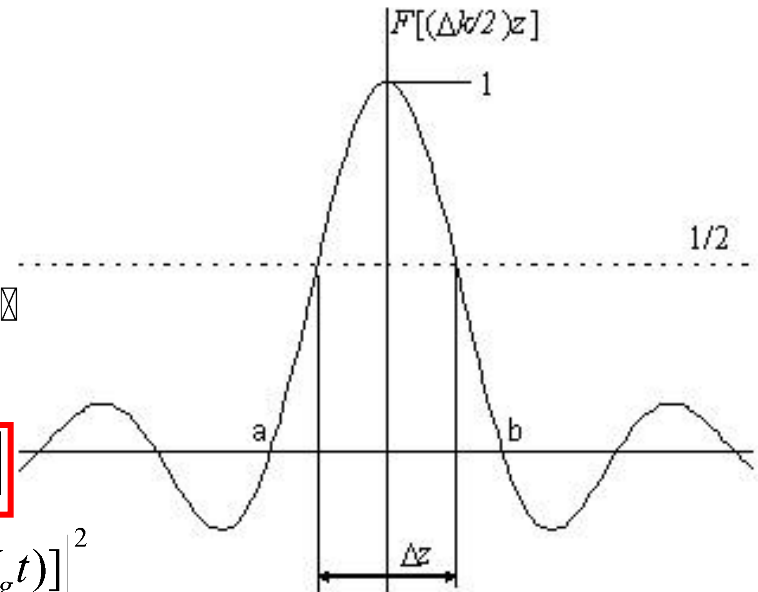
Примеры дельтаобразных *A*(*k*) и *A*(*z*)

$$A_0(k)=C_0\frac{2\pi}{\Delta k}\begin{cases}1, & k\in\left[k_0-\Delta k/2,\ k_0+\Delta k/2\right]\\ 0, & k\notin\left[k_0-\Delta k/2,\ k_0+\Delta k/2\right]\end{cases}$$

$$A_0(z)=F\left[(\Delta k/2)z\right]\cdot C_0\exp(ik_0z)\,,\quad F(\xi)=\frac{\sin(\xi)}{\xi}$$

$$A(t,z)=C_0\frac{1}{\Delta k}\int_{k_0-\Delta k/2}^{k_0+\Delta k/2}\exp\left[-i\omega(k)t+ikz\right]dk$$

$$\omega(k)=\omega(k_0)+(k-k_0)\left(\frac{d\omega}{dk}\right)_{k=k_0}+\xcancel{\frac{1}{2}(k-k_0)^2\left(\frac{d^2\omega}{dk^2}\right)_{k=k_0}}+\ldots$$

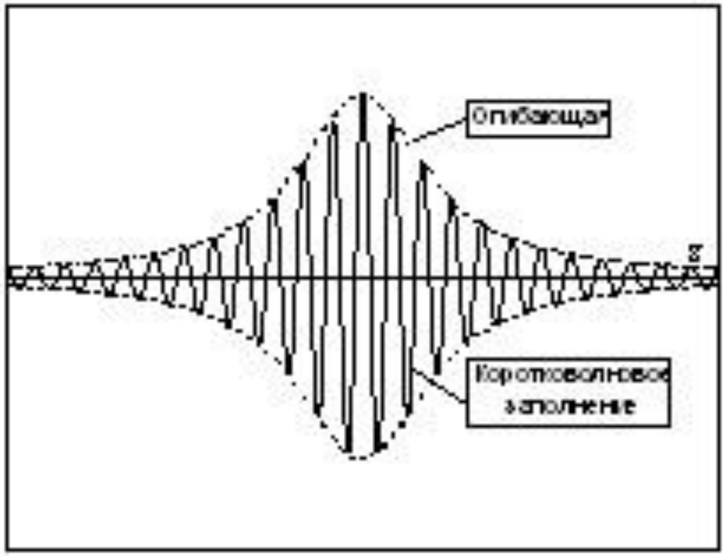


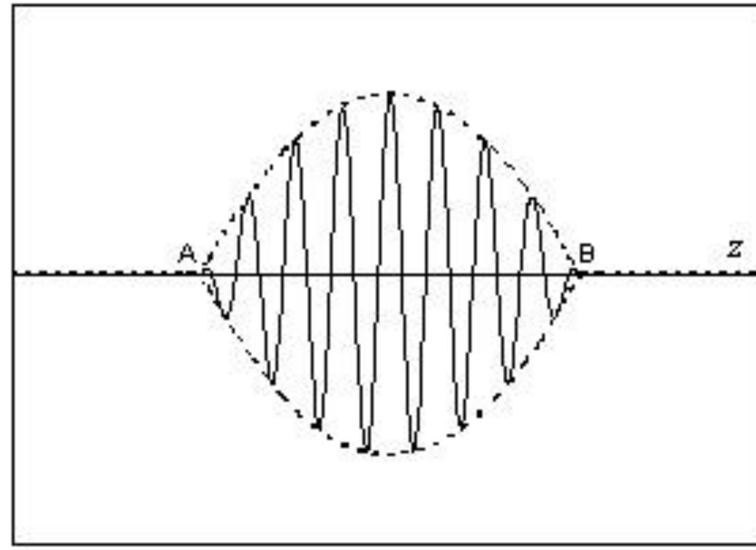
$$A(t,z)=F[(\Delta k/2)(z-V_gt)]\cdot C_0\exp\left[-i\omega(k_0)t+ik_0z\right]$$

$$V_g=\left(\frac{d\omega}{dk}\right)_{k=k_0}\qquad W(t,z)=|A(t,z)|^2=\left|C_0\frac{\sin[(\Delta k/2)(z-V_gt)]}{[(\Delta k/2)(z-V_gt)]}\right|^2$$

$$V_g=c_0\qquad V_g=\frac{c}{\sqrt{1+\omega_{Le}^2/k^2c^2}}=\frac{c^2}{V_\Phi}\qquad V_g=\frac{V_{Te}}{\sqrt{1+\omega_{Le}^2/(k^2V_{\gamma e}^2)}}=\frac{V_{\gamma e}^2}{V_\Phi}\qquad V_g=2kc^2\frac{\Omega_e}{\omega_p^2}=2V_\Phi$$

$$V_g=0\qquad V_g=\frac{c_0}{\sqrt{1+k_\perp^2/k^2}}=\frac{c_0^2}{V_\Phi}\qquad V_g=\frac{V_S}{(1+k^2r_{D_e}^2)^{3/2}}=\frac{V_\Phi}{(1+k^2r_{D_e}^2)}\qquad V_g^{(Б,М)}=u$$





$$A(t,z) = \Phi(t,z)\exp(-i\omega_0 t + ik_0 z)$$

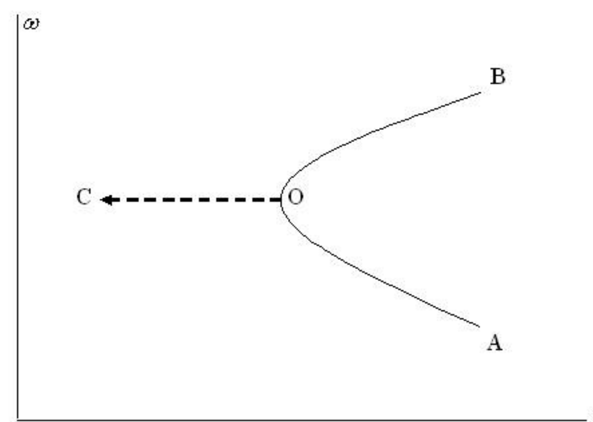
$$D\left(\omega_0 + i\frac{\partial}{\partial t}, k_0 - i\frac{\partial}{\partial z}\right)\Phi(t,z) = 0$$

$$\frac{\partial D}{\partial \omega}\frac{\partial \Phi}{\partial t} - \frac{\partial D}{\partial k}\frac{\partial \Phi}{\partial z} + i\Re = 0$$

$$V_g = \frac{d\omega}{dk} = -\frac{\partial D}{\partial k}\bigg/\frac{\partial D}{\partial \omega}$$

$$\frac{\partial \Phi}{\partial t} + V_g\frac{\partial \Phi}{\partial z} = 0$$

$$\frac{\partial D}{\partial \omega} = 0 \rightarrow \quad \omega(k) = \omega_0 \pm \left(2\frac{\partial D}{\partial k}\bigg/\frac{\partial^2 D}{\partial \omega^2}\right)^{1/2}\sqrt{k - k_0} + \dots$$



$$A(t,\mathbf{r}) = \tilde{A}(t,\mathbf{r})\exp(-i\omega_0 t + i\mathbf{k}_0\mathbf{r}) \qquad D\left(\omega_0 + i\frac{\partial}{\partial t}, \mathbf{k}_0 - i\frac{\partial}{\partial \mathbf{r}}\right)\tilde{A}(t,\mathbf{r}) = 0$$

$$\boxed{\frac{\partial \tilde{A}}{\partial t} + \mathbf{V}_g \frac{\partial \tilde{A}}{\partial \mathbf{r}} = 0} \qquad V_g = \frac{d\omega}{d\mathbf{k}} = -\frac{\partial D}{\partial \mathbf{k}} \Big/ \frac{\partial D}{\partial \omega} \qquad \boxed{\frac{\partial W}{\partial t} + \mathbf{V}_g \frac{\partial W}{\partial \mathbf{r}} = 0} \qquad W(t,\mathbf{r}) \sim |A(t,\mathbf{r})|^2$$

$$\boxed{\mathbf{S}(t,\mathbf{r}) = \mathbf{V}_g W(t,\mathbf{r})} \qquad \frac{\partial W}{\partial t} + \operatorname{div}\mathbf{S} = 0 \;\rightarrow\; \frac{d}{dt}\iiint_V W(t,\mathbf{r})d\mathbf{r} = -\iint_\Sigma \mathbf{S}\, d\boldsymbol{\sigma}$$

$$\boxed{\begin{aligned} \mathbf{S}_l(t,\mathbf{r}) &= -\frac{\partial}{\partial \mathbf{k}}[\omega\varepsilon^l(\omega,\mathbf{k})]\frac{|\tilde{E}_l(t,\mathbf{r})|^2}{16\pi} \\ \mathbf{S}_{tr}(t,\mathbf{r}) &= -\frac{\partial}{\partial \mathbf{k}}\left[\omega\left(\varepsilon^{tr}(\omega,\mathbf{k}) - \frac{k^2c^2}{\omega^2}\right)\right]\frac{|\tilde{E}_{tr}(t,\mathbf{r})|^2}{16\pi} \end{aligned}}$$

$$\mathbf{S}_{tr}(t,\mathbf{r}) = \frac{\mathbf{k}c^2}{\omega}\frac{|\tilde{E}_{tr}(t,\mathbf{r})|^2}{8\pi}$$

$$\Uparrow$$

$$c[\mathbf{E}\times\mathbf{B}]/4\pi$$

$$\varepsilon^l = \varepsilon^{tr} = 1 - \frac{\omega_{Le}^2}{(\omega^2 - k^2V_{\gamma e}^2)} \;\rightarrow$$

$$\mathbf{S}_l(t,\mathbf{r}) = \frac{\mathbf{k}}{\omega}V_{\gamma e}^2\frac{\omega^2}{\omega_{Le}^2}\frac{|\tilde{E}_l(t,\mathbf{r})|^2}{8\pi}$$

$$\mathbf{S}_{tr}(t,\mathbf{r}) = \frac{\mathbf{k}}{\omega}\left(c^2 + V_{\gamma e}^2\frac{\omega_{Le}^2\omega^2}{(\omega^2 - k^2V_{\gamma e}^2)}\right)\frac{|\tilde{E}_{tr}(t,\mathbf{r})|^2}{8\pi}$$

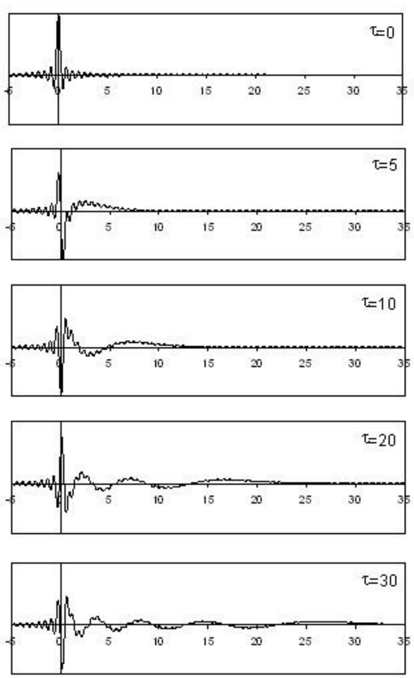
$$\Delta t \sim 2\pi \left((\Delta k)^2 \left| \frac{d^2\omega}{dk^2} \right| \right)^{-1}$$

$$\omega(k) = \frac{kV_0}{\sqrt{1 + k^2 V_0^2 / \Omega_0^2}}$$ – акустический тип

$$\left(\frac{\partial^2}{\partial t^2} - V_0^2 \frac{\partial^2}{\partial z^2} - V_0^2 \Omega_0^{-2} \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial z^2} \right) A(t,z) = 0$$

$$A_0(k) = C_0 \frac{\pi}{\Delta k} \begin{cases} 1, & k \in [-\Delta k,\ \Delta k], \\ 0, & k \notin [-\Delta k,\ \Delta k]. \end{cases}$$

$$a(\tau,\xi) = \frac{1}{\Delta} \int_0^{\Delta} \cos\left(\frac{x}{\sqrt{1+x^2}} \tau - x\xi \right) dx$$

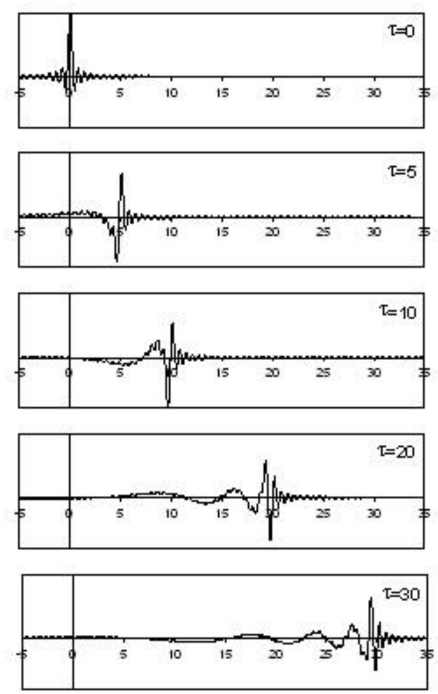


$$\omega(k) = \sqrt{\Omega_0^2 + k^2 V_0^2}$$ – оптический тип

$$\left(\frac{\partial^2}{\partial t^2} - V_0^2 \frac{\partial^2}{\partial z^2} + \Omega_0^2 \right) A(t,z) = 0$$

$$\tau = \Omega_0 t,\ \ \xi = \frac{\Omega_0}{V_0} z,\ \ x = \frac{kV_0}{\Omega_0},\ \ \Delta = (\Delta k) \frac{V_0}{\Omega_0}.$$

$$a(\tau,\xi) = \frac{1}{\Delta} \int_0^{\Delta} \cos\left(\sqrt{1+x^2}\,\tau - x\xi \right) dx$$



$$I(\lambda)=\int_a^b P(q)\exp[i\lambda\Psi(q)]\,dq \qquad x=\Psi(x)\to I(\lambda)=\int_{\Psi(a)}^{\Psi(b)}\left(P(x)/\Psi'(x)\right)\exp(i\lambda x)dx\sim\frac{1}{\lambda}$$

$$\Psi'(q_0)=0 \qquad P(q_0)\exp\left[i\lambda\left(\Psi(q_0)+\frac{1}{2}(q-q_0)^2\Psi''(q_0)\right)\right]$$

$$I(\lambda)=\left|\frac{2\pi}{\lambda\Psi''(q_0)}\right|^{1/2}P(q_0)\exp\left(i\lambda\Psi(q_0)\pm i\frac{\pi}{4}\right)$$

$$A(t,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}A_0(k)\exp[it\Psi(k)]\,dk, \qquad \Psi(k)=k\frac{z}{t}-\omega(k) \qquad \boxed{\frac{d\Psi}{dk}=\frac{z}{t}-V_g(k)=0}$$

$$k_0(\xi)=\frac{\Omega_0}{V_0}\left[\left(\frac{V_0}{\xi}\right)^{2/3}-1\right]^{1/2} \qquad k_0(\xi)=\frac{\Omega_0}{V_0}\left[\left(\frac{V_0}{\xi}\right)^{2}-1\right]^{-1/2} \qquad \xi=z/t$$

$$\boxed{A(t,z)=\frac{A_0(k_0)}{\sqrt{2\pi t\,|G(k_0)|}}\exp\left[-i\omega(k_0)t+ik_0z\pm i\frac{\pi}{4}\right], \quad G(k_0)=\left(\frac{d^2\omega}{dk^2}\right)_{k=k_0}, \quad k_0=k_0(z/t)}$$

$$\frac{dk_0}{dz}=\frac{1}{t\,|G(k_0)|}$$