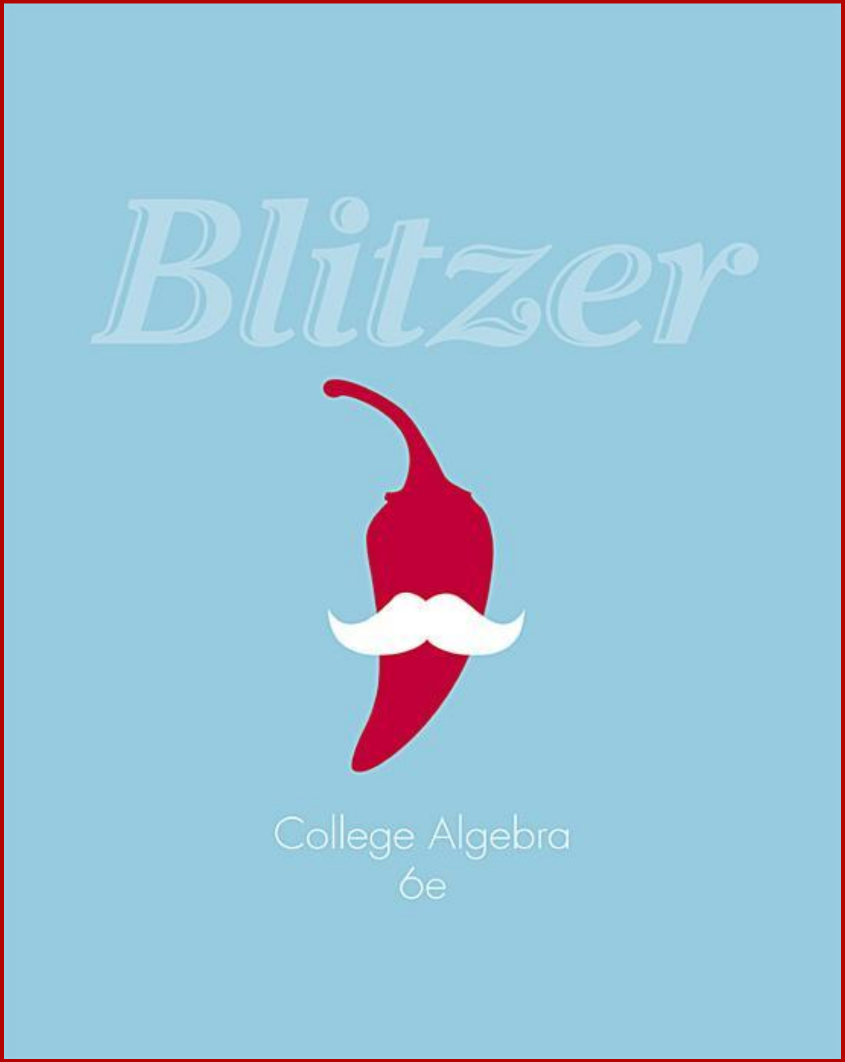
Chapter 3

Polynomial and Rational Functions

3.2 Polynomial Functions and Their Graphs



Objectives:

- Identify polynomial functions.
- Recognize characteristics of graphs of polynomial functions.
- Determine end behavior.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Use the Intermediate Value Theorem.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.

Definition of a Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers, with $a_n \neq 0$. The function defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

is called a polynomial function of degree n. The number a_n, the coefficient of the variable to the highest power, is called the leading coefficient.

Graphs of Polynomial Functions – Smooth and Continuous

Polynomial functions of degree 2 or higher have graphs that are *smooth* and *continuous*.

By **smooth**, we mean that the graphs contain only rounded curves with no sharp corners.

By **continuous**, we mean that the graphs have no breaks and can be drawn without lifting your pencil from the rectangular coordinate system.

End Behavior of Polynomial Functions

The **end behavior** of the graph of a function to the far left or the far right is called its end behavior.

Although the graph of a polynomial function may have intervals where it increases or decreases, the graph will eventually rise or fall without bound as it moves far to the left or far to the right.

The sign of the leading coefficient, a_n, and the degree, n, of the polynomial function reveal its end behavior.

The Leading Coefficient Test

As x increases or decreases without bound, the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

eventually rises or falls. In particular, the sign of the leading coefficient, a_n, and the degree, n, of the polynomial function reveal its end behavior.

The Leading Coefficient Test for $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ (continued)

1. For n odd:

If the leading coefficient is positive, the graph falls to the left and rises to the right. (↙, ↗)

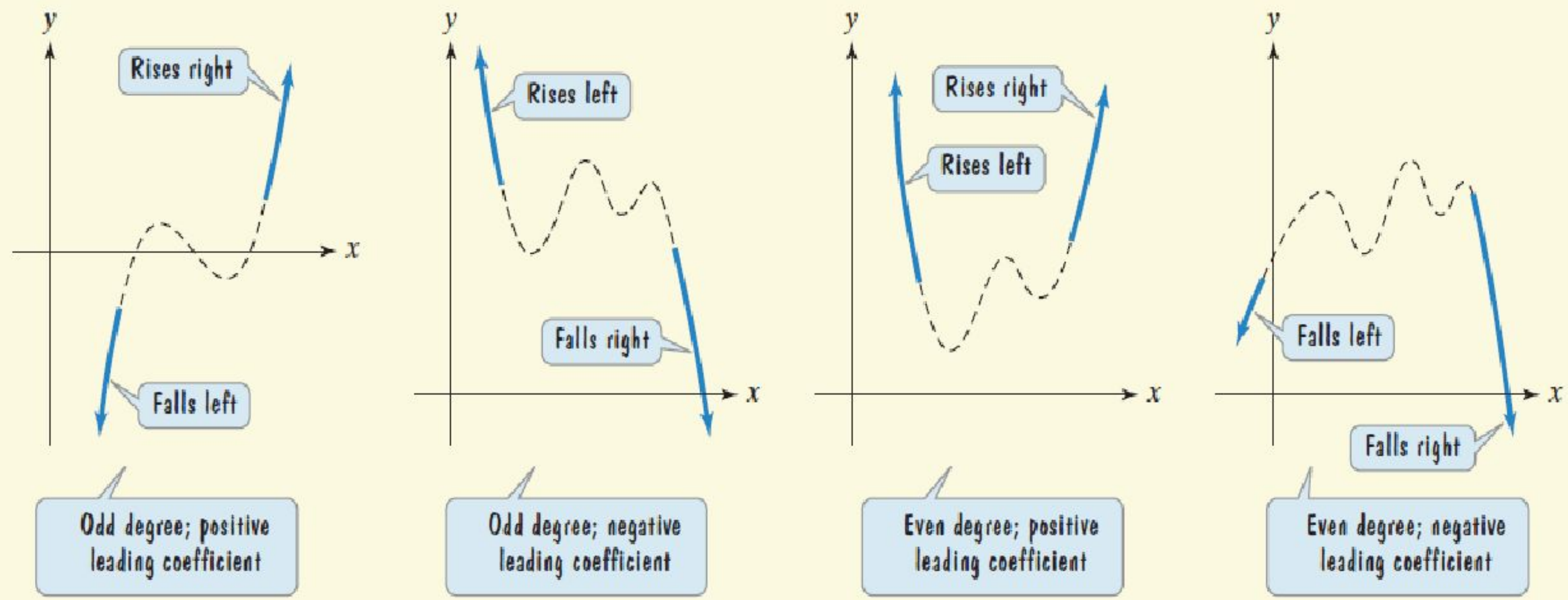
If the leading coefficient is negative, the graph rises to the left and falls to the right. (↖, ↘)

2. For n even:

If the leading coefficient is positive, the graph rises to the left and rises to the right. (↖, ↗)

If the leading coefficient is negative, the graph falls to the left and falls to the right. (↙, ↘)

$a_n > 0$ | $a_n < 0$ | $a_n > 0$ | $a_n < 0$

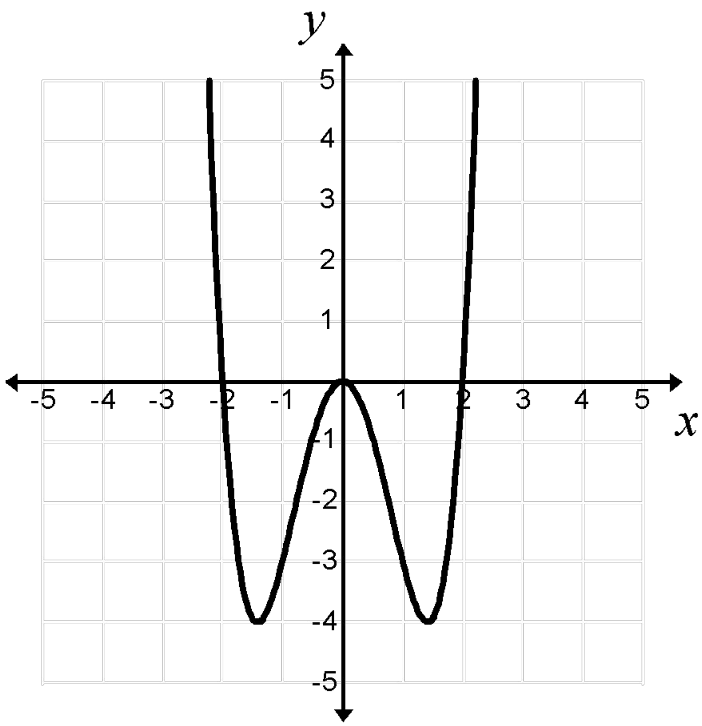


Example: Using the Leading Coefficient Test

Use the Leading Coefficient Test to determine the end behavior of the graph of $f(x) = x^4 - 4x^2$

The degree of the function is 4, which is even. Even-degree functions have graphs with the same behavior at each end.

The leading coefficient, 1, is positive. The graph rises to the left and to the right.



Zeros of Polynomial Functions

If f is a polynomial function, then the values of x for which $f(x)$ is equal to 0 are called the **zeros** of f.

These values of x are the **roots**, or **solutions**, of the polynomial equation $f(x) = 0$.

Each real root of the polynomial equation appears as an x-intercept of the graph of the polynomial function.

Example: Finding Zeros of a Polynomial Function

Find all zeros of $f(x) = x^3 + 2x^2 - 4x - 8$

We find the zeros of *f* by setting *f*(*x*) equal to 0 and solving the resulting equation.

$$f(x) = x^3 + 2x^2 - 4x - 8$$
$$0 = x^3 + 2x^2 - 4x - 8$$
$$0 = x^2(x+2) - 4(x+2)$$
$$0 = (x+2)(x^2 - 4)$$

$(x+2) = 0$ or $(x^2 - 4) = 0$

$$x + 2 = 0$$
$$x = -2$$

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm\sqrt{4} = \pm 2$$

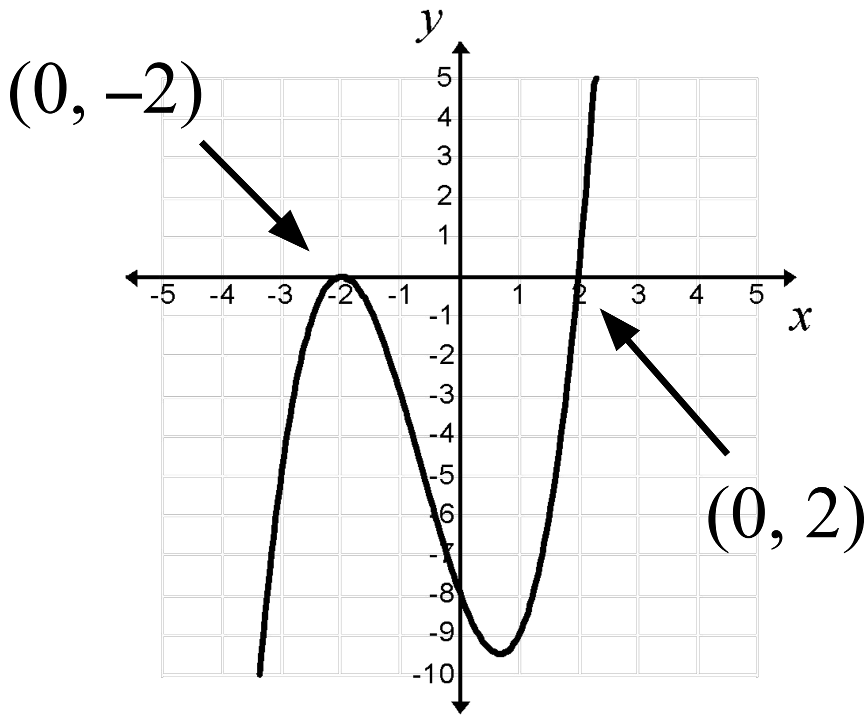
Example: Finding Zeros of a Polynomial Function *(continued)*

Find all zeros of $f(x) = x^3 + 2x^2 - 4x - 8$

The zeros of *f* are –2 and 2.

The graph of f shows that each zero is an *x*-intercept.

The graph passes through (0, –2) and (0, 2).



Multiplicity and x-Intercepts

If r is a zero of **even multiplicity**, then the graph **touches** the x-axis and turns around at r. If r is a zero of **odd multiplicity**, then the graph **crosses** the x-axis at r. Regardless of whether the multiplicity of a zero is even or odd, graphs tend to flatten out near zeros with multiplicity greater than one.

Example: Finding Zeros and Their Multiplicities

Find the zeros of $f(x) = -4\left(x+\frac{1}{2}\right)^2 (x-5)^3$

and give the multiplicities of each zero. State whether the graph crosses the x-axis or touches the x-axis and turns around at each zero.

Example: Finding Zeros and Their Multiplicities *(continued)*

We find the zeros of f by setting $f(x)$ equal to 0:

$$f(x) = -4\left(x+\frac{1}{2}\right)^2 (x-5)^3 \rightarrow -4\left(x+\frac{1}{2}\right)^2 (x-5)^3 = 0$$

$$\left(x+\frac{1}{2}\right)^2 = 0 \rightarrow x = -\frac{1}{2}$$

$$(x-5)^3 = 0 \rightarrow x = 5$$

$x = -\frac{1}{2}$ is a zero of multiplicity 2.

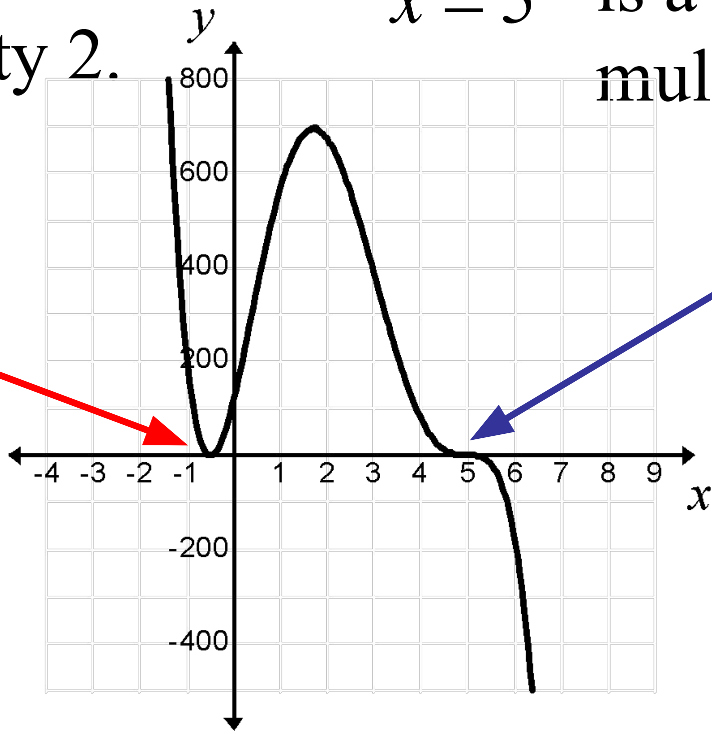
$x = 5$ is a zero of multiplicity 3.

Example: Finding Zeros and Their Multiplicities *(continued)*

For the function $f(x) = -4\left(x + \frac{1}{2}\right)^2 (x-5)^3$

$x = -\frac{1}{2}$ is a zero of multiplicity 2.

$x = 5$ is a zero of multiplicity 3.



The graph will touch the x-axis at $x = -\frac{1}{2}$

The graph will cross the x-axis at $x = 5$

The Intermediate Value Theorem

Let f be a polynomial function with real coefficients. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of c between a and b for which $f(c) = 0$. Equivalently, the equation $f(x) = 0$ has at least one real root between a and b.

Example: Using the Intermediate Value Theorem

Show that the polynomial function $f(x) = 3x^3 - 10x + 9$ has a real zero between –3 and –2.

We evaluate *f* at –3 and –2. If *f*(–3) and *f*(–2) have opposite signs, then there is at least one real zero between –3 and –2.

$$f(x) = 3x^3 - 10x + 9$$

$$f(-3) = 3(-3)^3 - 10(-3) + 9 = -81 + 30 + 9 = -42$$

$$f(-2) = 3(-2)^3 - 10(-2) + 9 = -24 + 20 + 9 = 5$$

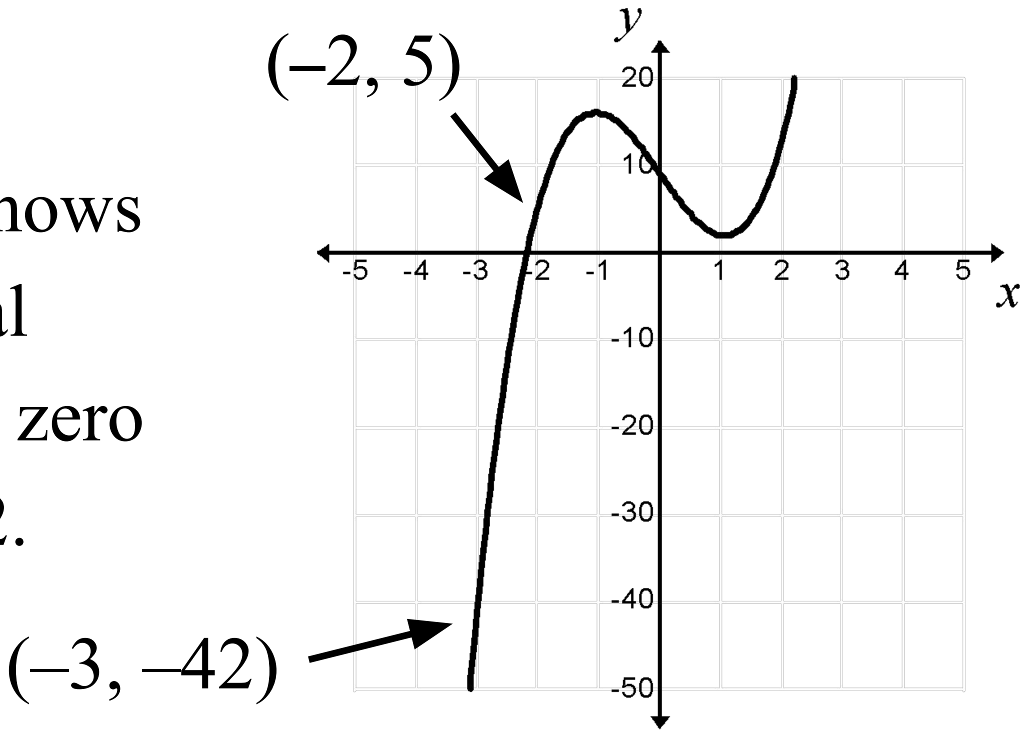
Example: Using the Intermediate Value Theorem *(continued)*

For $f(x) = 3x^3 - 10x + 9,$

$f(-3) = -42$

and $f(-2) = 5.$

The sign change shows that the polynomial function has a real zero between –3 and –2.



Turning Points of Polynomial Functions

In general, **if *f* is a polynomial function of degree *n*, then the graph of *f* has at most $n - 1$ turning points.**

A Strategy for Graphing Polynomial Functions

Graphing a Polynomial Function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0, a_n \neq 0$$

1. Use the Leading Coefficient Test to determine the graph's end behavior.
2. Find x-intercepts by setting $f(x) = 0$ and solving the resulting polynomial equation. If there is an x-intercept at r as a result of $(x - r)^k$ in the complete factorization of $f(x)$, then
   - **a.** If k is even, the graph touches the x-axis at r and turns around.
   - **b.** If k is odd, the graph crosses the x-axis at r.
   - **c.** If $k > 1$, the graph flattens out near $(r, 0)$.
3. Find the y-intercept by computing $f(0)$.
4. Use symmetry, if applicable, to help draw the graph:
   - **a.** y-axis symmetry: $f(-x) = f(x)$
   - **b.** Origin symmetry: $f(-x) = -f(x)$.
5. Use the fact that the maximum number of turning points of the graph is $n - 1$, where n is the degree of the polynomial function, to check whether it is drawn correctly.

Example: Graphing a Polynomial Function

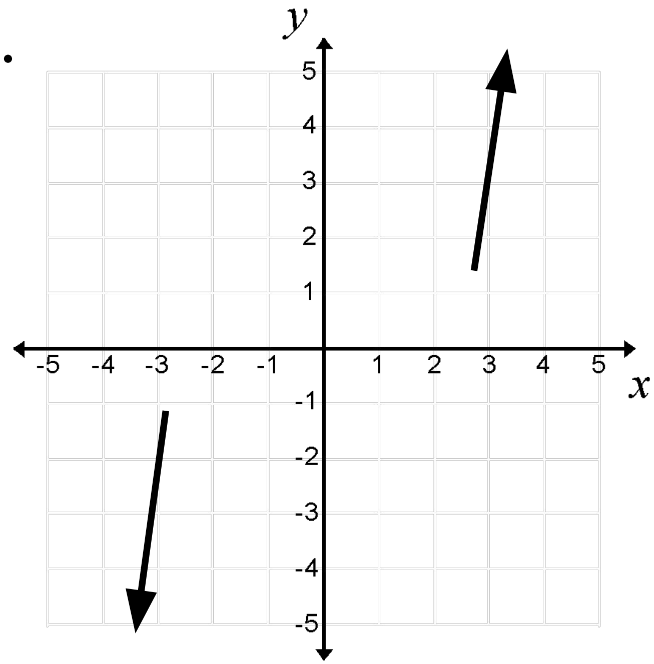
Use the five-step strategy to graph $f(x) = 2(x+2)^2(x-3)$

Step 1 Determine end behavior

Identify the sign of a_n, the leading coefficient, and the degree, n, of the polynomial function.

$a_n = 2$ and $n = 3$

The degree, 3, is odd. The leading coefficient, 2, is a positive number. The graph will rise on the right and fall on the left.



Example: Graphing a Polynomial Function *(continued)*

Use the five-step strategy to graph $f(x) = 2(x+2)^2(x-3)$

Step 2 Find *x*-intercepts (zeros of the function) by setting *f*(*x*) = 0.

$$f(x) = 2(x+2)^2(x-3)$$

$$2(x+2)^2(x-3) = 0$$

$$x+2=0 \rightarrow x=-2$$

$$x-3=0 \rightarrow x=3$$

$x = -2$ is a zero of multiplicity 2.

$x = 3$ is a zero of multiplicity 1.

Example: Graphing a Polynomial Function *(continued)*

Use the five-step strategy to graph $f(x) = 2(x+2)^2(x-3)$

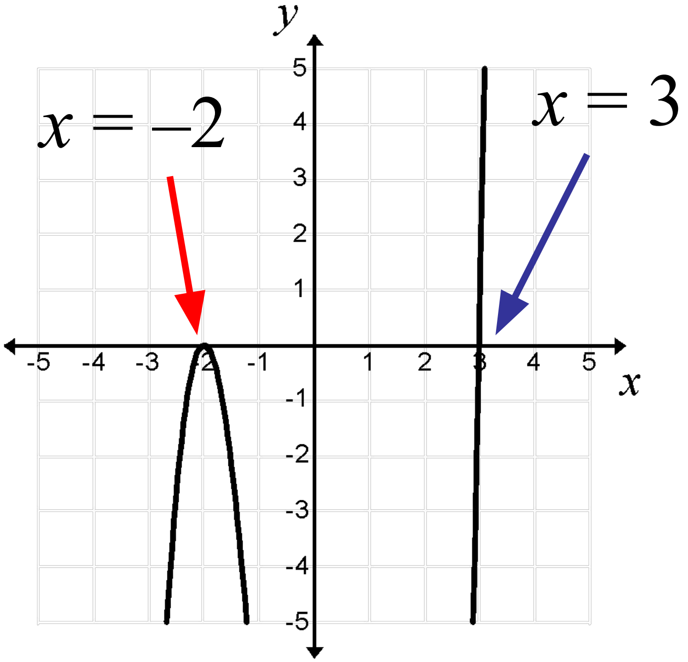
Step 2 *(continued)* Find *x*-intercepts (zeros of the function) by setting *f*(*x*) = 0.

$x = -2$ is a zero of multiplicity 2.

The graph touches the *x*-axis at $x = -2$, flattens and turns around.

$x = 3$ is a zero of multiplicity 1.

The graph crosses the *x*-axis at $x = 3$.



Example: Graphing a Polynomial Function *(continued)*

Use the five-step strategy to graph $f(x) = 2(x+2)^2(x-3)$

Step 3 Find the *y*-intercept by computing *f*(0).

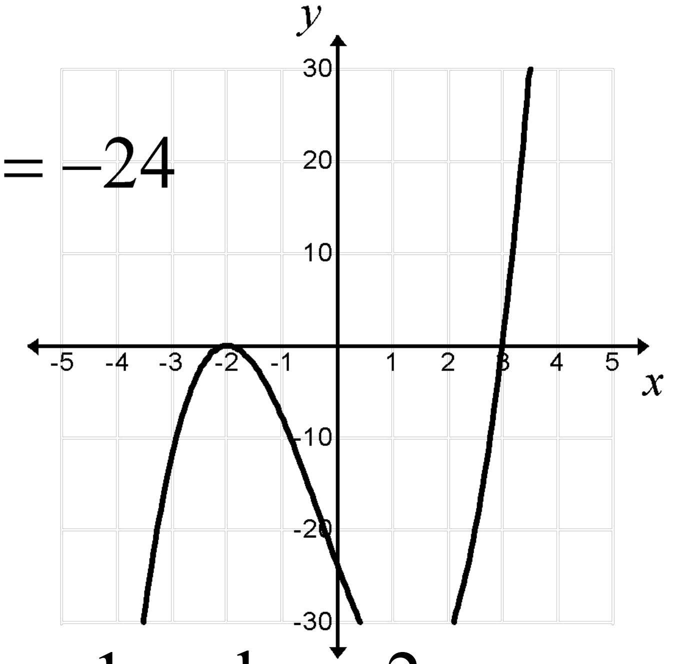
$f(x) = 2(x+2)^2(x-3)$

$f(0) = 2(0+2)^2(0-3) = 2(4)(-3) = -24$

The *y*-intercept is –24.

The graph passes through the *y*-axis at (0, –24).

To help us determine how to scale the graph, we will evaluate *f*(*x*) at *x* = 1 and *x* = 2.



Example: Graphing a Polynomial Function *(continued)*

Use the five-step strategy to graph $f(x) = 2(x+2)^2(x-3)$

Step 3 ***(continued)*** **Find the *y*-intercept by computing *f*(0).**

The *y*-intercept is –24. The graph passes through the *y*-axis at (0, –24). To help us determine how to scale the graph, we will evaluate *f*(*x*) at *x* = 1 and *x* = 2.

$$f(x) = 2(x+2)^2(x-3)$$

$$f(1) = 2(1+2)^2(1-3) = 2(9)(-2) = -36$$

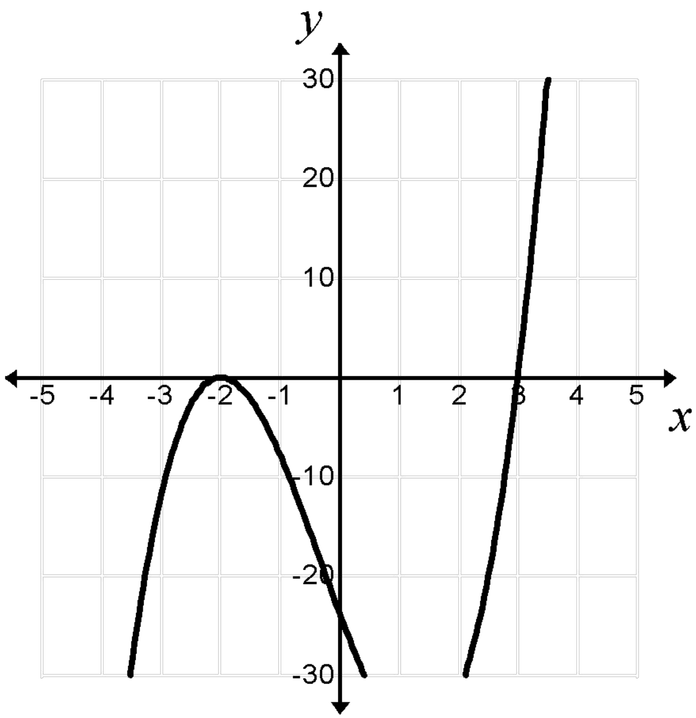
$$f(2) = 2(2+2)^2(2-3) = 2(16)(-1) = -32$$

Example: Graphing a Polynomial Function *(continued)*

Use the five-step strategy to graph $f(x) = 2(x+2)^2(x-3)$

Step 4 Use possible symmetry to help draw the graph.

Our partial graph illustrates that we have neither *y*-axis symmetry nor origin symmetry.

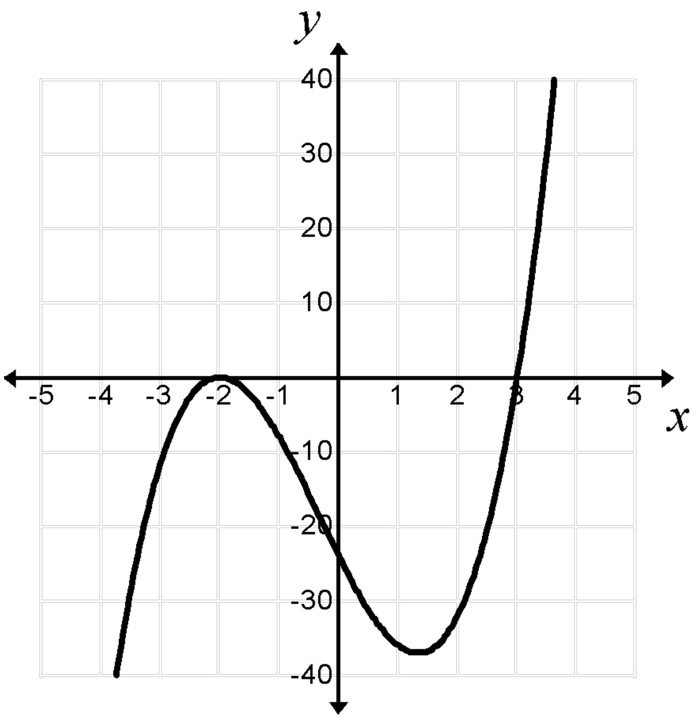


Example: Graphing a Polynomial Function *(continued)*

Use the five-step strategy to graph $f(x) = 2(x+2)^2(x-3)$

Step 4 *(continued)* Use possible symmetry to help draw the graph.

Our partial graph illustrated that we have neither *y*-axis symmetry nor origin symmetry. Using end behavior, intercepts, and the additional points, we graph the function.



Example: Graphing a Polynomial Function *(continued)*

Use the five-step strategy to graph $f(x) = 2(x+2)^2(x-3)$

Step 5 Use the fact that the maximum number of turning points of the graph is *n*-1 to check whether it is drawn correctly.

The degree is 3. The maximum number of turning points will be 3 – 1 or 2. Because the graph has two turning points, we have not violated the maximum number possible.