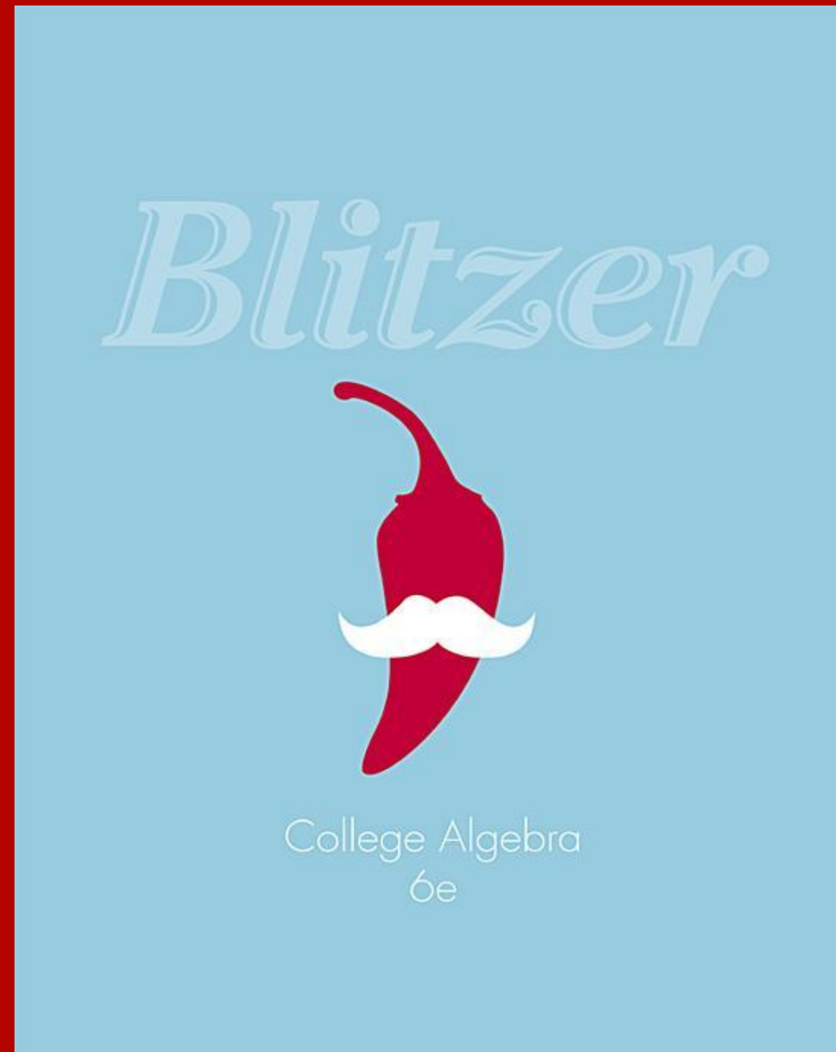
Chapter 2

Functions and Graphs

2.6 Combinations of Functions; Composite Functions



Objectives:

- Find the domain of a function.
- Combine functions using the algebra of functions, specifying domains.
- Form composite functions.
- Determine domains for composite functions.
- Write functions as compositions.

Finding a Function's Domain

If a function f does not model data or verbal conditions, its domain is the largest set of real numbers for which the value of $f(x)$ is a real number. Exclude from a function's domain real numbers that cause division by zero and real numbers that result in a square root of a negative number.

Example: Finding the Domain of a Function

Find the domain of the function $g(x)=\dfrac{5x}{x^2-49}$

Because division by 0 is undefined, we must exclude from the domain the values of x that cause the denominator to equal zero.

$$x^2-49=0$$
$$x^2=49$$
$$x=\pm\sqrt{49}$$
$$x=\pm 7$$

We exclude 7 and – 7 from the domain of g.

The domain of g is

$$(-\infty,-7)\cup(-7,7)\cup(7,\infty)$$

The Algebra of Functions: Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions. The sum $f + g$, the difference, $f - g$, the product fg, and the quotient $\frac{f}{g}$ are functions whose domains are the set of all real numbers common to the domains of f and g $(D_f \cap D_g)$, defined as follows:

1. Sum: $(f+g)(x) = f(x) + g(x)$
2. Difference: $(f-g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Example: Combining Functions

Let $f(x) = x - 5$ and $g(x) = x^2 - 1$. Find each of the following:

a. $(f+g)(x) = (x-5)+(x^2-1) = x^2 + x - 6$

b. The domain of $(f+g)(x)$

The domain of $f(x)$ has no restrictions.

The domain of $g(x)$ has no restrictions.

The domain of $(f+g)(x)$ is $(-\infty, \infty)$

The Composition of Functions

The **composition of the function *f* with *g*** is denoted $f \circ g$ and is defined by the equation

$$(f \circ g)(x) = f(g(x))$$

The **domain of the composite function** $f \circ g$ is the set of all x such that

1. x is in the domain of g and
2. $g(x)$ is in the domain of f.

Example: Forming Composite Functions

Given $f(x) = 5x + 6$ and $g(x) = 2x^2 - x - 1$, find $f \circ g$

$$
\begin{aligned}
f \circ g = f(g(x)) &= f(2x^2 - x - 1) \\
&= 5(2x^2 - x - 1) + 6 \\
&= 10x^2 - 5x - 5 + 6 \\
&= 10x^2 - 5x + 1
\end{aligned}
$$

Excluding Values from the Domain of $(f \circ g)(x) = f(g(x))$

The following values must be excluded from the input x:

If x is not in the domain of g, it must not be in the domain of $f \circ g$.

Any x for which $g(x)$ is not in the domain of f must not be in the domain of $f \circ g$.

Example: Forming a Composite Function and Finding Its Domain

Given $f(x) = \dfrac{4}{x+2}$ and $g(x) = \dfrac{1}{x}$

Find $(f \circ g)(x)$

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \frac{4}{\frac{1}{x}+2} = \frac{4}{\frac{1}{x}+2} \cdot \frac{x}{x}$$

$$(f \circ g)(x) = f(g(x)) = \frac{4x}{1+2x}$$

Example: Forming a Composite Function and Finding Its Domain

Given $f(x) = \dfrac{4}{x+2}$ and $g(x) = \dfrac{1}{x}$

Find the domain of $(f \circ g)(x)$

For $g(x)$, $x \neq 0$

For $(f \circ g)(x) = \dfrac{4x}{1+2x}$, $x \neq -\dfrac{1}{2}$

The domain of $(f \circ g)(x)$ is $\left(-\infty, -\dfrac{1}{2}\right) \cup \left(-\dfrac{1}{2}, 0\right) \cup (0, \infty)$

Example: Writing a Function as a Composition

Express $h(x)$ as a composition of two functions:

$$h(x) = \sqrt{x^2 + 5}$$

If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 5,$ then $h(x) = (f \circ g)(x)$