LECTURE 3

MIXED STRATEGY NASH EQUILIBRIUM

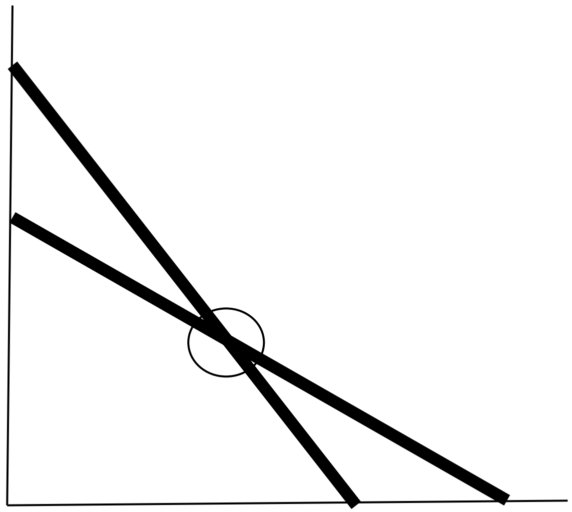
Review

- The **Nash equilibrium** is the likely outcome of simultaneous games, both for discrete and continuous sets of actions.
  - Derive the best response functions, find where they intersect.
- We have considered NE where players select one action with probability 100% □ **Pure strategies**
  - For each action of the Player 2, the best response of Player 1 is a deterministic (i.e. non random) action
  - For each action of the Player 1, the best response of Player 2 is a deterministic action

Review

- A Nash equilibrium in which every player plays a pure strategy is a **pure strategy Nash equilibrium**
  - At the equilibrium, each player plays only one action with probability 1.

	No Ad	Ad
No Ad	50 , 50	20 , 60
Ad	60 , 20	30 , 30



Overview

- Pure strategy NE is just one type of NE, another type is mixed strategy NE.
  - A player plays a mixed strategy when he chooses randomly between several actions.
- Some games do not have a pure strategy NE, but have a mixed strategy NE.
- Other games have both pure strategy NE and mixed strategy NE.

Employee Monitoring

- Consider a company where:
  - Employees can work hard or shirk
    - Salary: $100K unless caught shirking
    - Cost of effort: $50K
  - The manager can monitor or not
    - An employee caught shirking is fired
    - Value of employee output: $200K
    - Profit if employee doesn't work: $0
    - Cost of monitoring: $10K

"Wake up Harper, it's five o'clock - I don't want you putting in for overtime."

Employee Monitoring

Manager

100-50 200-100-10 200-100

Employee		Monitor	No monitor
	Work	50,90	50,100
	Shirk	0, -10	100,-100

0-10 0-100

- No equilibrium in pure strategies
- What is the likely outcome?

Football penalty shooting

Football penalty shooting

Goal Keeper


KICKER


	L	R
L	-1 , 1	1 , -1
R	1 , -1	-1 , 1

Football penalty shooting

- No equilibrium in pure strategies
  - Similar to the employee/manager game
- How would you play this game?
  - Players must make their actions unpredictable
- Suppose that the goal keeper jumps left with probability p, and jumps right with probability 1-p.
  - What is the kicker’s best response?

Football penalty shooting

- If p=1, i.e. if goal keeper always jumps left
  - then we should kick right
- If p=0, i.e. if goal keeper always jumps right
  - then we should kick left

- The kicker's expected payoff is:
  - $\pi(\text{left})$: $-1 \times p + 1 \times (1-p) = 1 - 2p$
  - $\pi(\text{right})$: $1 \times p - 1 \times (1-p) = 2p - 1$

- $\pi(\text{left}) > \pi(\text{right})$ if $p<1/2$

Football penalty shooting

- Should kick left if: $p < ½$
  $(1 – 2p > 2p – 1)$
- Should kick right if: $p > ½$
- Is indifferent if: $p = ½$
- What value of p is best for the goal keeper?

Keeper's p	Kicker's strategy	Keeper's Payoff
L (p = 1)	R	-1.0
R (p = 0)	L	-1.0
p = 0.75	R	-0.5
p = 0.55	R	-0.1
p = 0.50	Either	0

¼* 1- ¾ *1

0.45* 1-0.55 *1

Football penalty shooting

- Mixed strategy:
  - It makes sense for the goal keeper and the kicker to randomize their actions.
  - If opponent knows what I will do, I will always lose!
  - Players try to make themselves unpredictable.
- Implications:
  - A player chooses his strategy so as to prevent his opponent from having a winning strategy.
  - The opponent has to be made indifferent between his possible actions.

Employee Monitoring

Manager

Employee

		Monitor	No monitor
		q	1-q
Work	1-p	50,90	50,100
Shirk	p	0, -10	100,-100

- Employee chooses (shirk, work) with probabilities (p,1-p)
- Manager chooses (monitor, no monitor) with probabilities (q,1-q)

Keeping Employees from Shirking

- First, find employee's expected payoff from each pure strategy

- If employee works: receives 50
  - $\pi(\text{work}) = 50\times q + 50\times (1-q) = 50$

- If employee shirks: receives 0 or 100
  - $\pi(\text{shirk}) = 0\times q + 100\times(1-q)$

    $= 100 - 100q$

Employee's Best Response

- Next, calculate the best strategy for possible strategies of the opponent

- For **$q<1/2$**: **SHIRK**

  π (shirk) = 100-100q > 50 = π (work)
- For **$q>1/2$**: **WORK**

  π (shirk) = 100-100q < 50 = π (work)
- For **$q=1/2$**: **INDIFFERENT**

  π (shirk) = 100-100q = 50 = π (work)

The manager has to monitor just often enough to make the employee work (q=1/2). No need to monitor more than that.

Manager's Best Response

- Manager's payoff:
  - Monitor: $90\times(1-p)- 10\times p=90-100p$
  - No monitor: $100\times(1-p)-100\times p=100-200p$
- For **$p<1/10$**: **NO MONITOR**

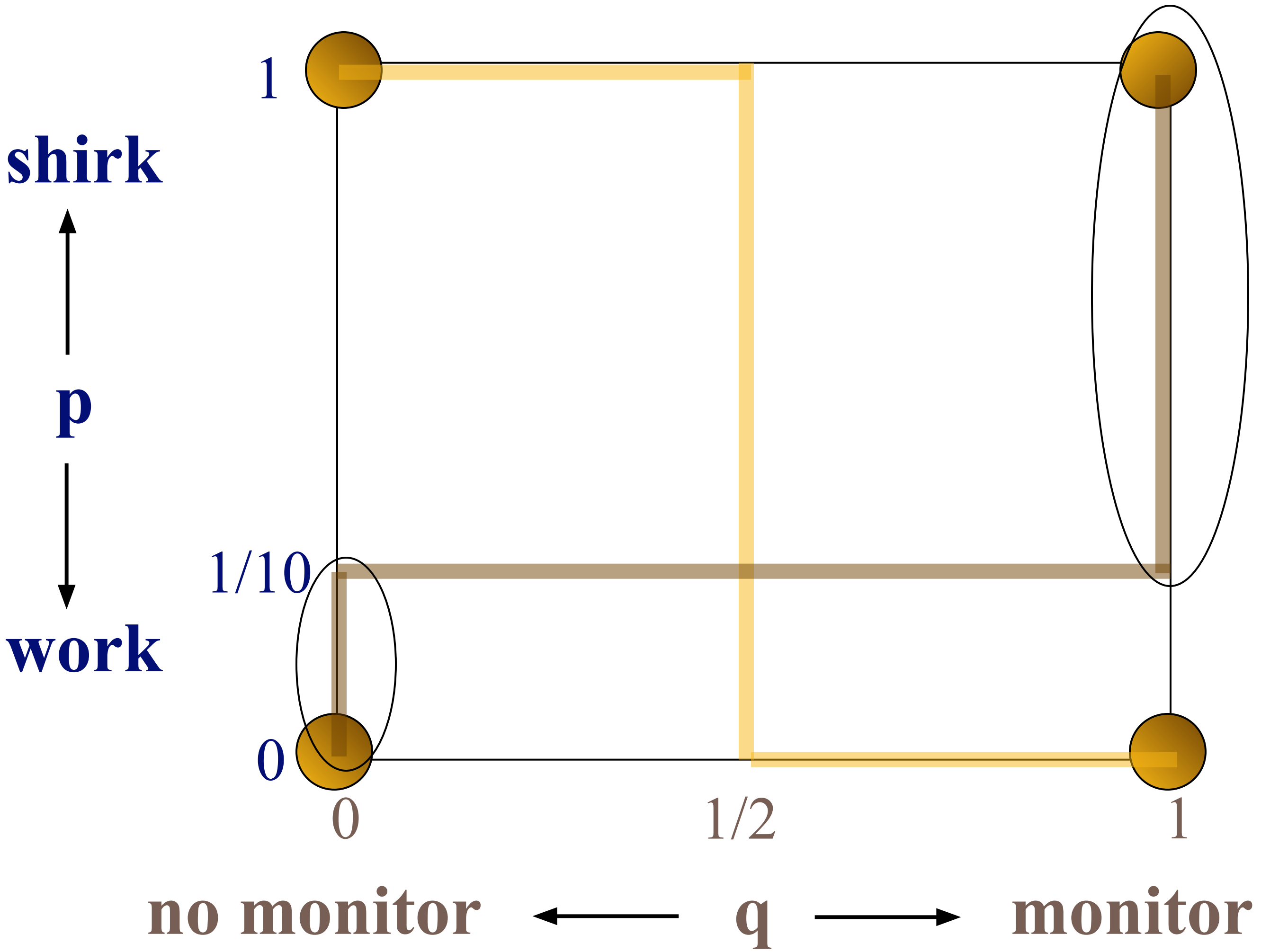
  $\pi(\text{monitor}) = 90-100p < 100-200p = \pi(\text{no monitor})$
- For **$p>1/10$**: **MONITOR**

  $\pi(\text{monitor}) = 90-100p > 100-200p = \pi(\text{no monitor})$
- For **$p=1/10$**: **INDIFFERENT**

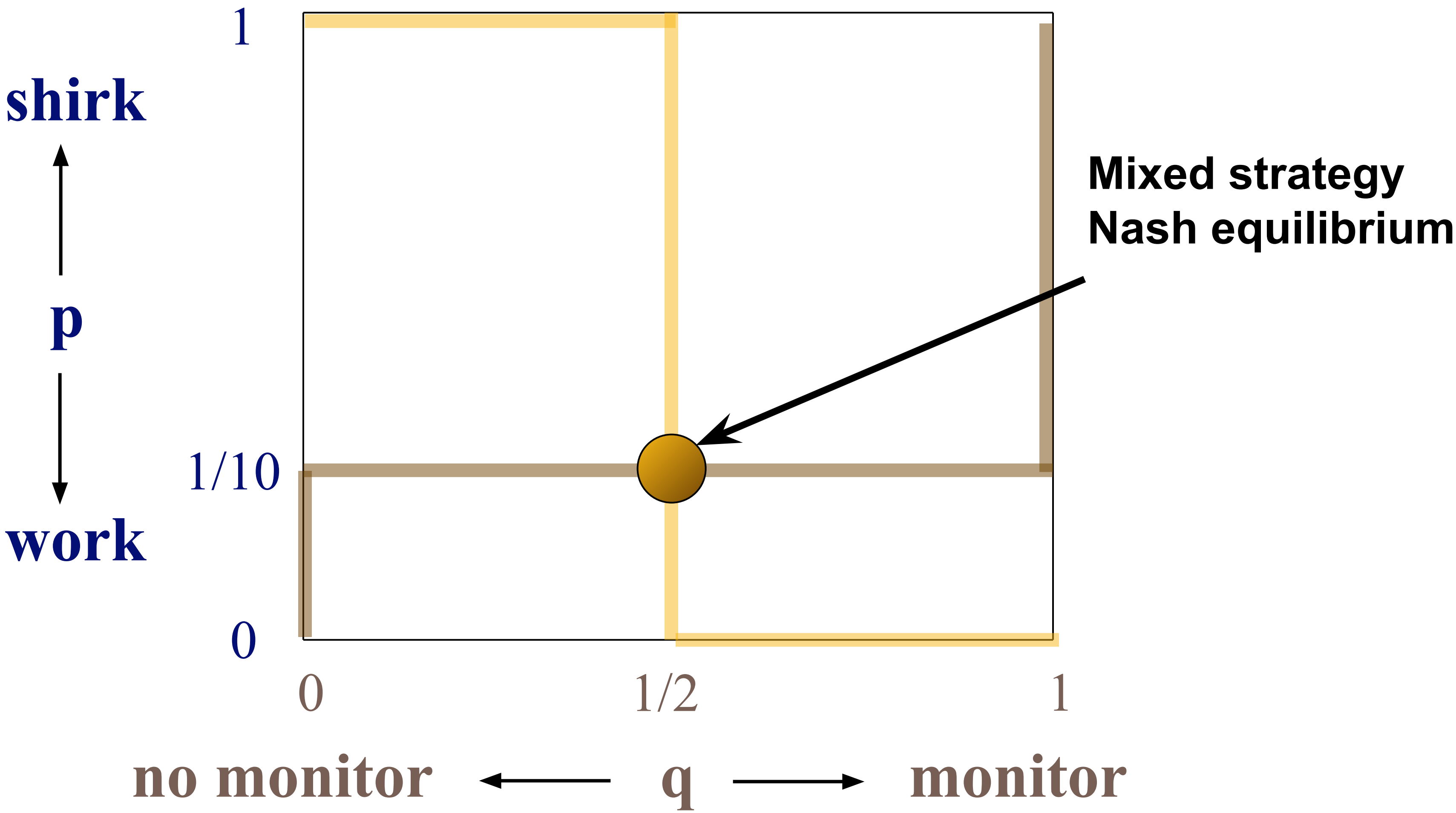
  $\pi(\text{monitor}) = 90-100p = 100-200p = \pi(\text{no monitor})$

The employee has to work just enough to make the manager not monitor ($p=1/10$). No need to work more than that.

Best responses



Mutual Best Responses



Equilibrium strategies

Manager

Employee

		Monitor	No monitor
		50%	50%
Work	90%	50,90	50,100
Shirk	10%	0, -10	100,-100

At the equilibrium, both players are indifferent between the two possible strategies.

Equilibrium payoffs

- Employee
  - π (shirk)=0+100x0.5=50
  - π (work)=50
- Manager
  - π (monitor)=0.9x90-0.1x10=80
  - π (no monitor)=0.9x100-0.1x100=80

Theorems

1. If there are no pure strategy equilibria, there must be a unique mixed strategy equilibrium.

2. However, it is possible for pure strategy and mixed strategy Nash equilibria to coexist. (for example coordination games)

New Scenario

- What if cost of monitoring is 50, (instead of 10)?

Manager

Employee

	Monitor	No monitor
Work	50,50	50,100
Shirk	0,-50	100,-100

New Scenario

- To make employee indifferent:

  π(work)= π(shirk) implies

  $50=100 - 100q$

  $\Rightarrow q=1/2$

- To make manager indifferent

  π(monitor)= π(no monitor) implies

  $50-100p = 100-200p$

  $\Rightarrow p=1/2$

New Scenario

- Equilibrium:
  - q=1/2, unchanged
  - p=1/2, instead of 1/10
- Why does q remain unchanged?
  - Payoff of "shirk" unchanged: the manager must maintain a 50% probability of monitoring to prevent shirking.
    - If q=49%, employees always shirk.
- Cost of monitoring higher, thus employees can afford to shirk more.
- **One player's equilibrium mixture probabilities depend only on the other player's payoff**

Application: Tax audits

- Mix strategy to prevent tax evasion:
  - Random audits, just enough to induce people to pay their taxes.
- In 2002, IRS Commissioner noticed that:
  - Audits have become more expensive
  - Number of audits decreased slightly
  - Offshore evasion increased by $70 billion dollars
- Recommendation:

  As audits get more expensive, need to increase budget to keep number of audits constant!

Do players select the MSNE?

Mixed strategies in football

- Economist Palacios-Huerta analyzed 1,417 penalty kicks. Success matrix:

Goalie

Kicker			Left	Right
			q	1-q
	Left	p	58, 42	95, 5
	Right	1-p	93, 7	70, 30

- Equilibrium:
  - Kicker: $58q+95(1-q)=93q+70(1-q)$ □ **q=42%**
  - Goalie: $42p+7(1-p)=5p+30(1-p)$ □ **p=38%**

Do players select the MSNE?

Mixed strategies in football

- Observed behavior for the 1,417 penalty kicks:
  - Kickers choose left with probability 40%
    - Prediction was 38%
  - Goalies jump to the left with probability 42%
    - Prediction was 42%

- Players have the ability to randomize!

Entry

Coordination game

- Two firms are deciding which new market to enter. Market A is more profitable than market B

Firm 2

Firm 1

		A	B
		q	1-q
A	p	2,2	4,3
B	1-p	3,4	1,1

- Coordination game: 2 PSNE, where players enter a different market.

Entry

Coordination game

- Both player prefer choosing market A and let the other player choose market B.
  - Two PSNE.
- Expected payoff for Firm 1 when playing A

  $\pi(A)=2q+4(1-q)=4-2q$
- If it plays B:

  $\pi(B)=3q+(1-q)=1+2q$
- $\pi(A)= \pi(B)$ if $q=3/4$

Entry

Coordination game

- For Firm 2:

  $\pi(A) = \pi(B)$ □ $p=3/4$

- Equilibrium in mixed strategies: $p=q=3/4$
- Expected payoff:
  - Firm 1: $2\frac{9}{16}+4\frac{3}{16}+3\frac{3}{16}+1\frac{1}{16}=2.5$
  - Same for Firm 2.
- Expected payoff is 2.5 for both firms
  - Lower than 3 or 4 □In this example, pure strategy NE yields a higher payoff. There is a risk of miscoordination where both firms choose the same market.

In what types of games are mixed strategies most useful?

- For games of cooperation, there is 1 PSNE, and no MSNE.
- For games with no PSNE (e.g. shirk/monitor game), there is one MSNE, which is the most likely outcome.
- For coordination games (e.g. the entry game), there are 2 PSNE and 1 MSNE.
  - Theoretically, all equilibria are possible outcomes, but the difference in expected payoff may induce players to coordinate.

Weak sense of equilibrium

- Mixed strategy NE are NE in a weak sense
  - Players have no incentive to change action, but they would not be worse off if they did
    - $\pi(\text{shirk}) = \pi(\text{work})$
  - Why should a player choose the equilibrium mixture when the other one is choosing his own?

What Random Means

- Study
  - A fifteen percent chance of being stopped at an alcohol checkpoint will deter drinking and driving
- Implementation
  - Set up checkpoints one day a week ($1 / 7 \approx 14\%$)
  - How about Fridays?

Use the mixed strategy that keeps your opponents guessing.

BUT

Your probability of each action must be the same period to period.

Summary

- Games may not have a PSNE, and mixed strategies help predict the likely outcome in those situations, e.g. shirk/monitor game.
- Mixed strategies are also relevant in games with multiple PSNE, e.g. coordination games.
- Randomization. Make the other player indifferent between his strategies.