Ilya Yu. Kolesnikov

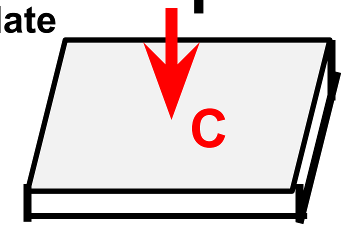
Robust Non-Algebraic Reissner-Mindlin Plate Finite Elements

Geophysical Center of the RAS

Study Subject : Locking VS. Stability for R – M thin plates

Over-Stiff FEM equations - Much slow convergence and poor accuracy

Low-order Algebraic Interpolation



W_C Kirchhoff exact = 0.01160000 Pa^2 / D

a priori independent

Compatible

LARGE → Stiff → W = small

	N = 6 x 6	N = (6 x 6)^2
h = 0.01	w = 0.00028325	w = 0.00534994
h = 0.001	w = 0.00000290	w = 0.00009847
h = 0.0001	w = 0.00000003	w = 0.00000099
h = 0.00001	w = 0.00000000	w = 0.00000001

Stable

Uniform

[2 x 2] Gauss – Legendre / Exact integration

0.00000

[2 x 2] + [1 x 1]

Rank Deficiency

0.01192

' Lim { 3D ; R - M } ' = ' Kirchhoff model

exact asymptotic

Morgenstern, 1959; Gol'denveizer, 1965; Babuska & Pitkaranta, 1990.

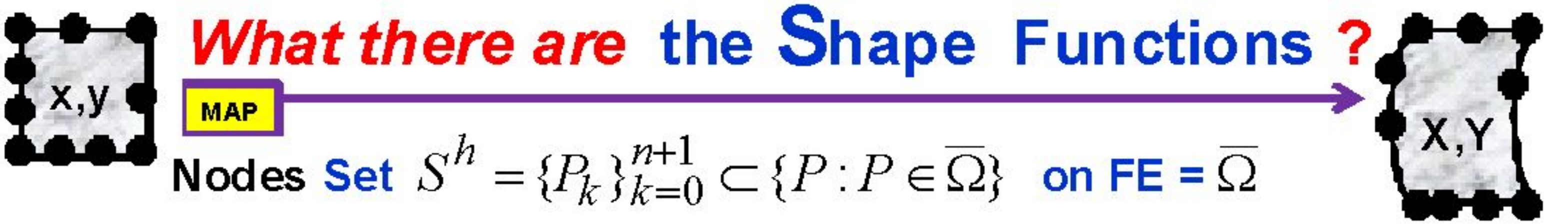
First

Reduced / Selective numerical integration, *Zienkiewicz* et el., 1971, 1976.

***A/S* rule: "*Accuracy* x Stability = *Constant*"**

The most "STATIC" area in FEM is Shape Functions of Algebraic type.

What there are the Shape Functions ?



Nodes Set $S^h = \{P_k\}_{k=0}^{n+1} \subset \{P : P \in \overline{\Omega}\}$ on FE = $\overline{\Omega}$

SF: basis Functions $\{f_j^h(P \in \overline{\Omega}; P_j \in S^h) \in C(\overline{\Omega})\}_{j=0}^{n+1}$ in $R_{C(\overline{\Omega})}^{n+2}$

Kronecker delta $f_j^h(P_k \in S^h; P_j \in S^h) = \delta_{jk}(j,k = 0,1,\ldots,n+1)$

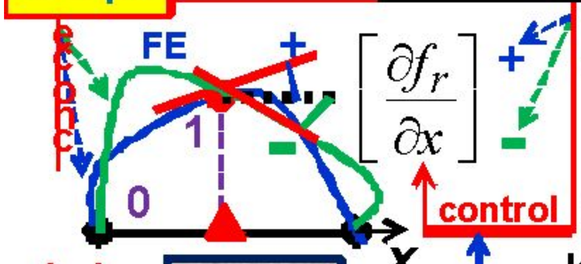
FE – *approximation* / interpolation $u^h(P) = \Sigma_{j=0}^{M+1} f_j^h(P; P_j \in S^h) u^h(P_j)$

Coordinates $M \leq n$ | *Physical* DoFs $M = n$

Partition of *Unity* $\Sigma_{j=0}^{n+1} f_j^h(P; P_j \in S^h) \equiv 1, \forall P \in \overline{\Omega}$

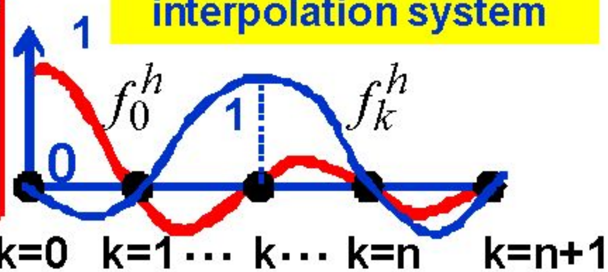
T – system $\{f_j^h(P; \{P_k\}_{k=0}^{n+1}) : (n+1)\ Zeroes\}_{j=0}^{n+1}$

Shape stiffness matrix block



choice | Numerical integration

Lagrange fundamental interpolation system



Strategy

FEM → FE → SF

SF construction

FOR ⇨ control

stiffness matrix

Solution Quality

NO Internal Nodes

1D, 2D, 3D Non – Algebraic Shape Functions for Arbitrary Number of Boundary Nodes

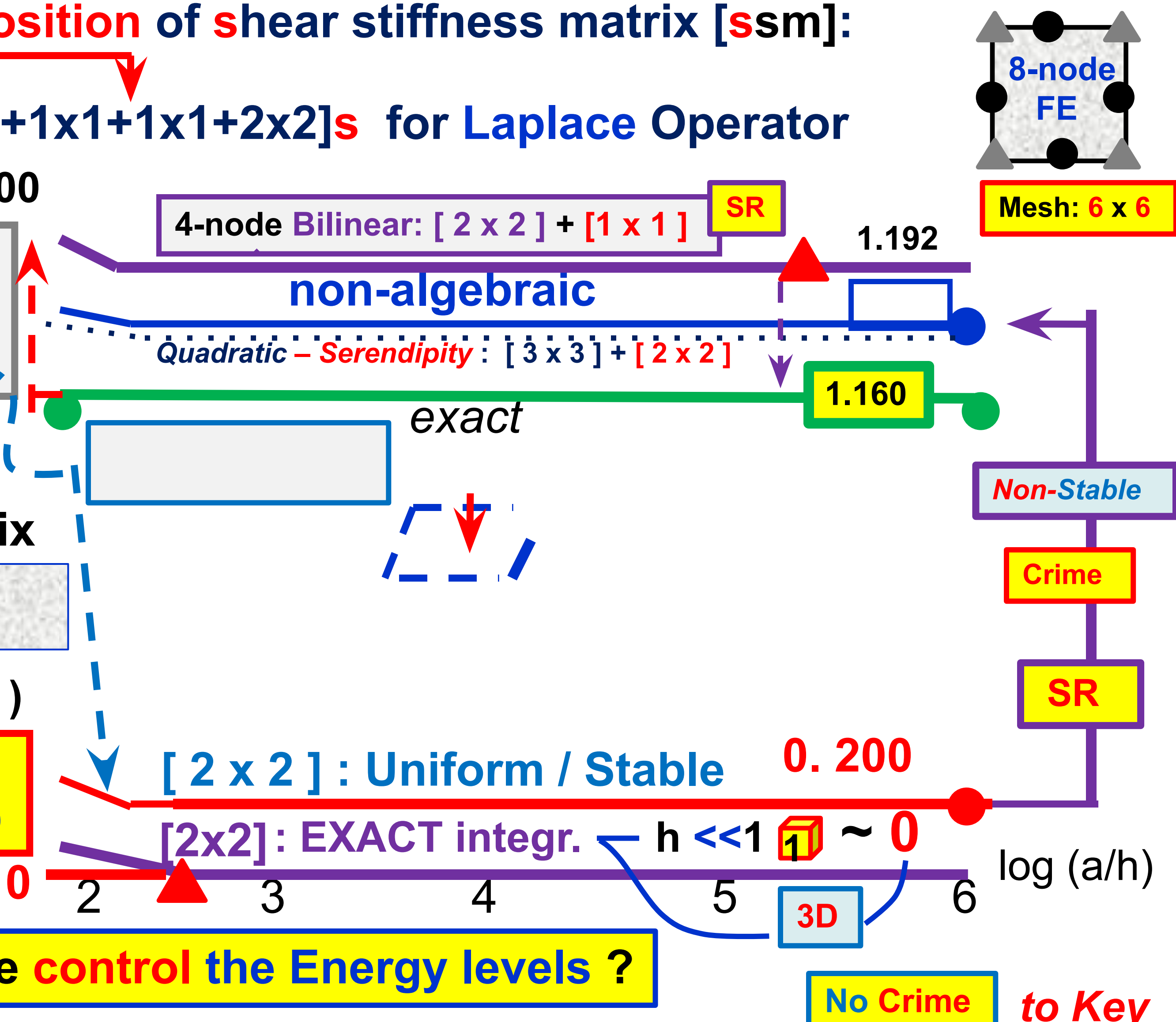
R – M shear locking problem with DoF:

S1: Scheme of Selective–Reduced Integration (SR)

with decomposition of shear stiffness matrix [ssm]:

Need

[2x2]b+[1x1+1x1+1x1+2x2]s for Laplace Operator



Field Inconsistency & Excessive-Stiffness = Delayed Convergence

element stiffness matrix

(Ex.: max =)

Only ONE level of the Energy (displacement)

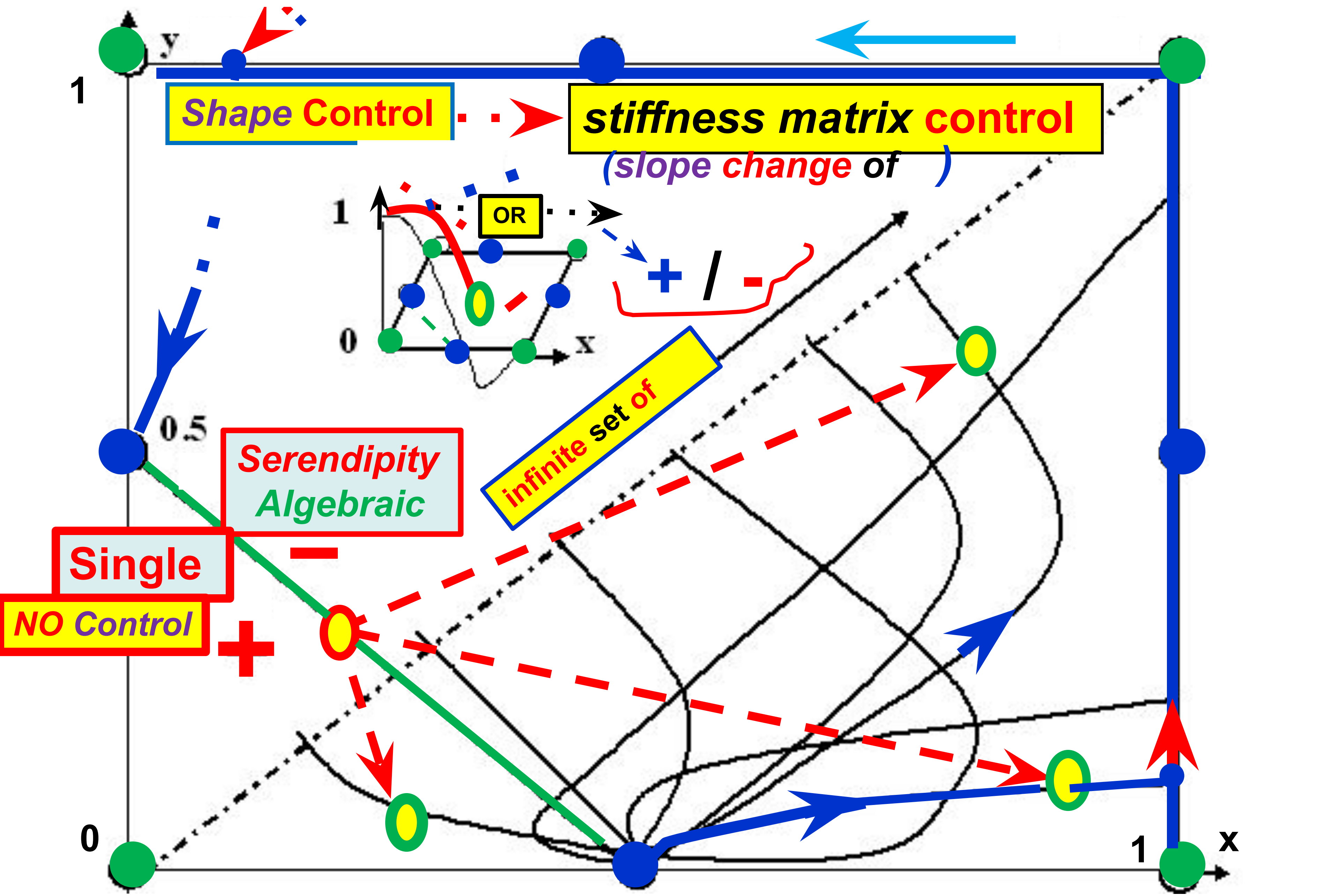
another way

How can we control the Energy levels ?

***to get Accurate/Stable solution:* Uniform int. [2 x 2]**

to Key

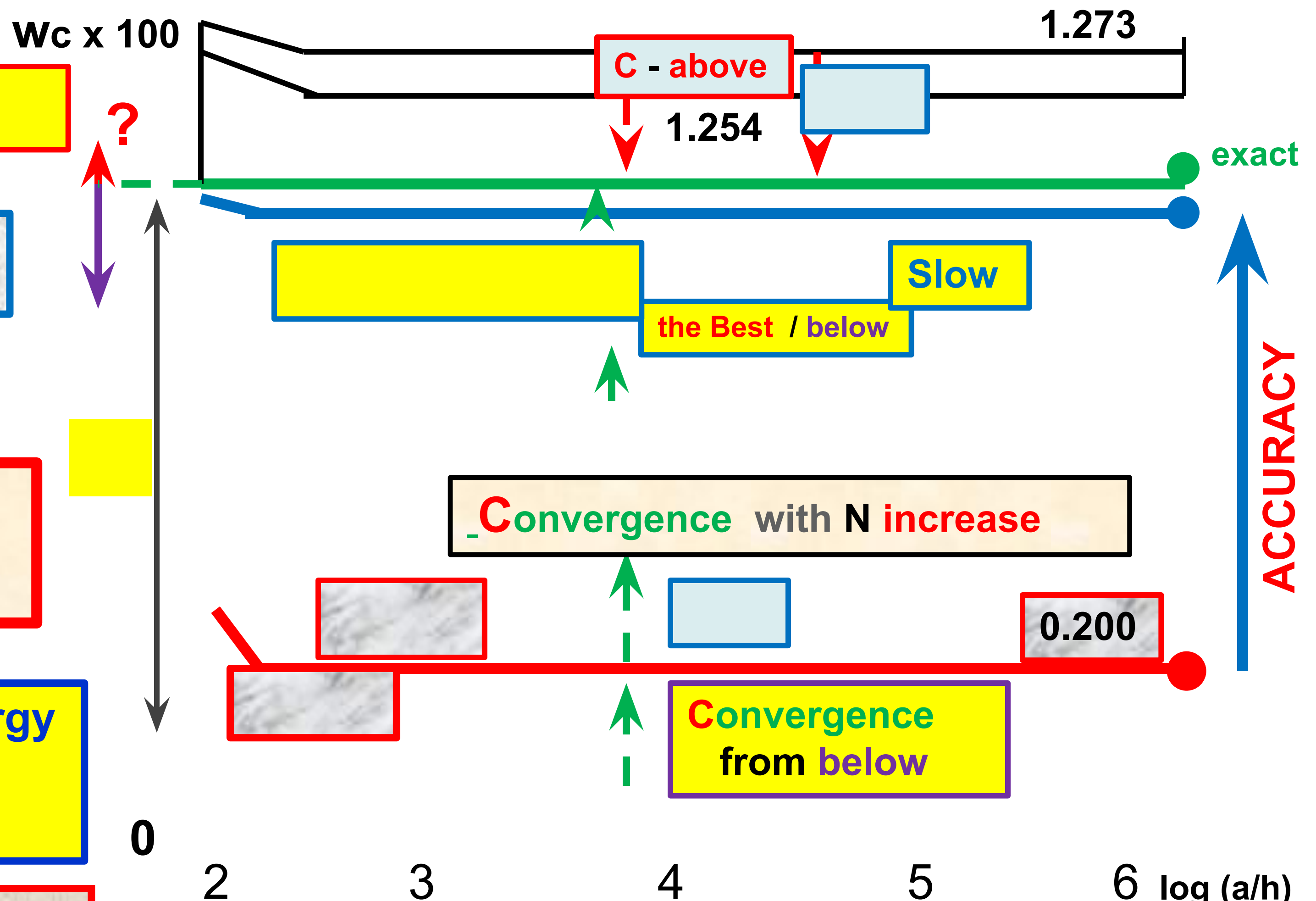
introduce - Shape Functions Variability



A Continuation Method: D.F. Davidenko; V.I. Shalashilin

Convergence Improvement

: Scheme of Full (Uniform) Integration (FI):
[2x2] for Helmholtz Operator



NO introducing
Rank Deficiency &
ill-Conditioning

Infinite Set of Energy
(displacement)
levels of

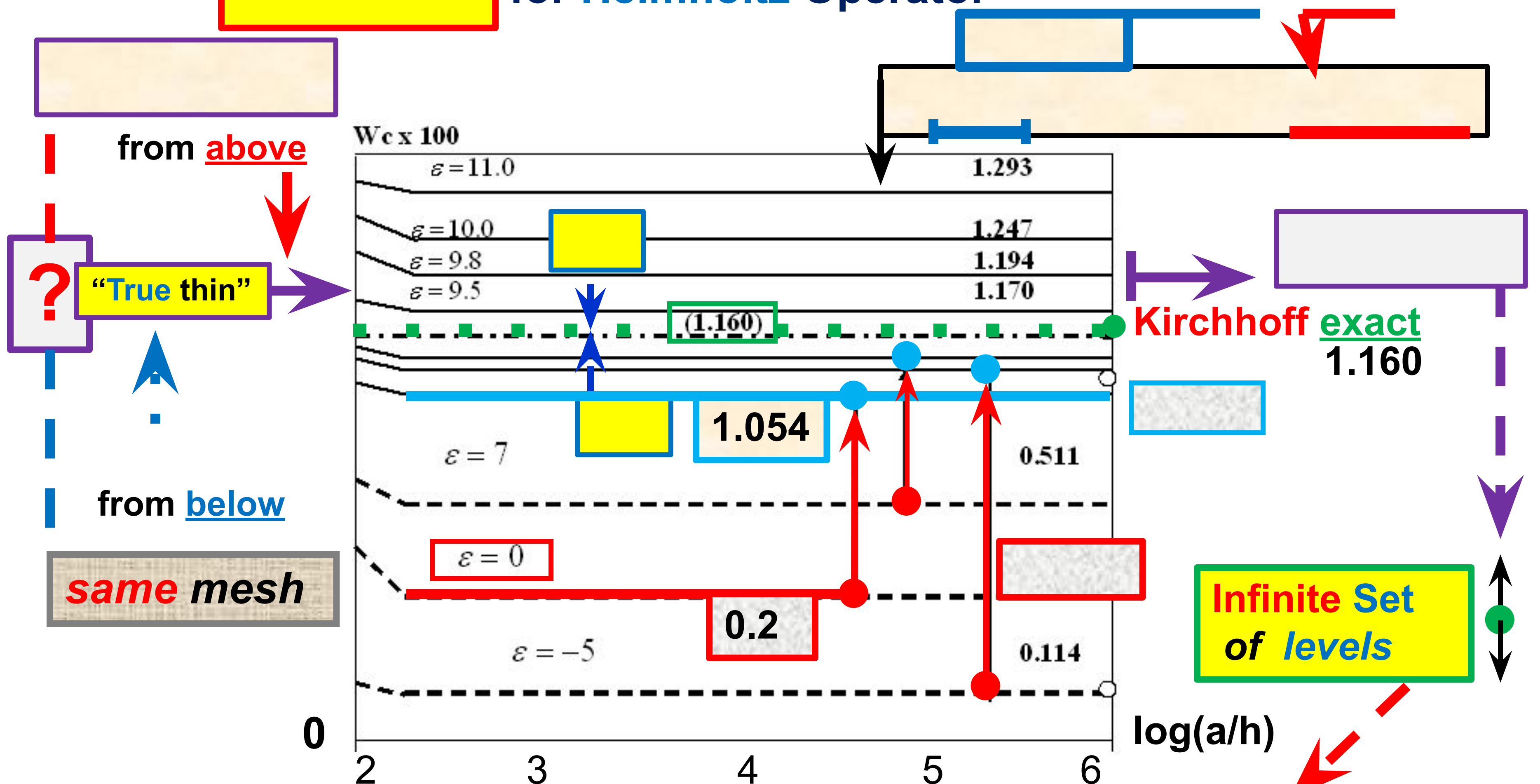
same coarse mesh

Q.: How we can MORE improve solution?

: **Multi-Scale Scheme: Slow (w) & Fast**

Full (Uniform) Integration (MS): [2x2]

for Helmholtz Operator



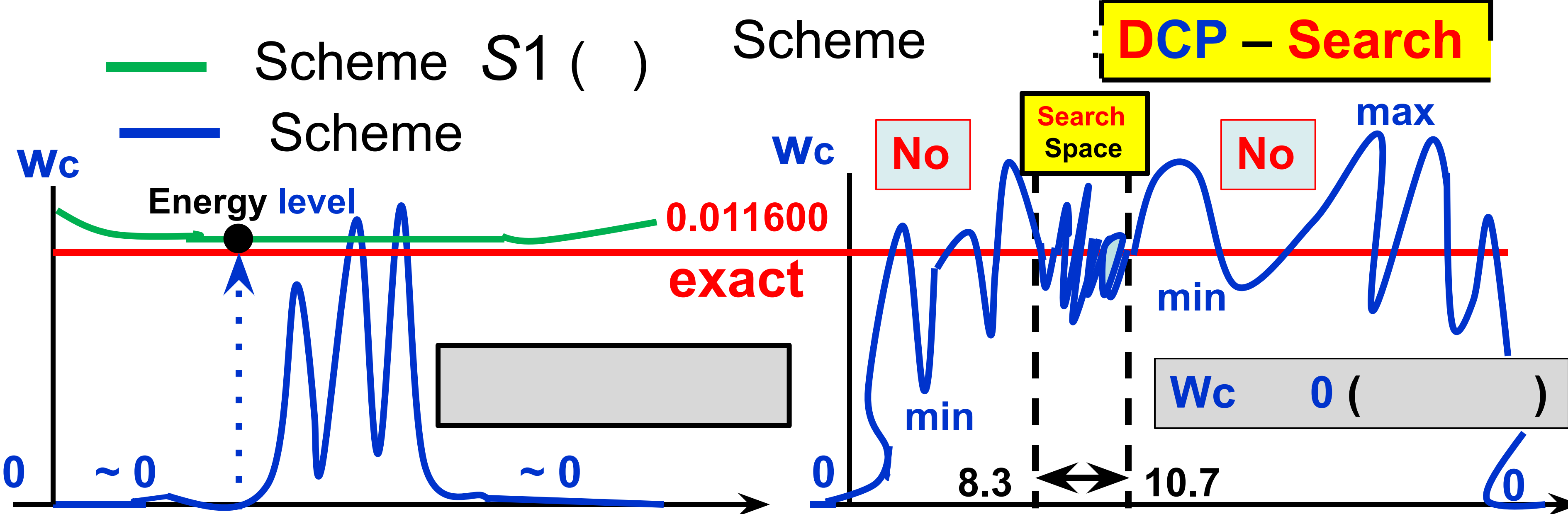
Scheme Better than Scheme

to Single level Selection

- Parametric Study

towards choice of Unique & Stable solution

Infinite Set of Energy (displacement) levels of



No DCP

may be

DCP – Degenerated / inflexion Critical Point : Structural Stability of Set

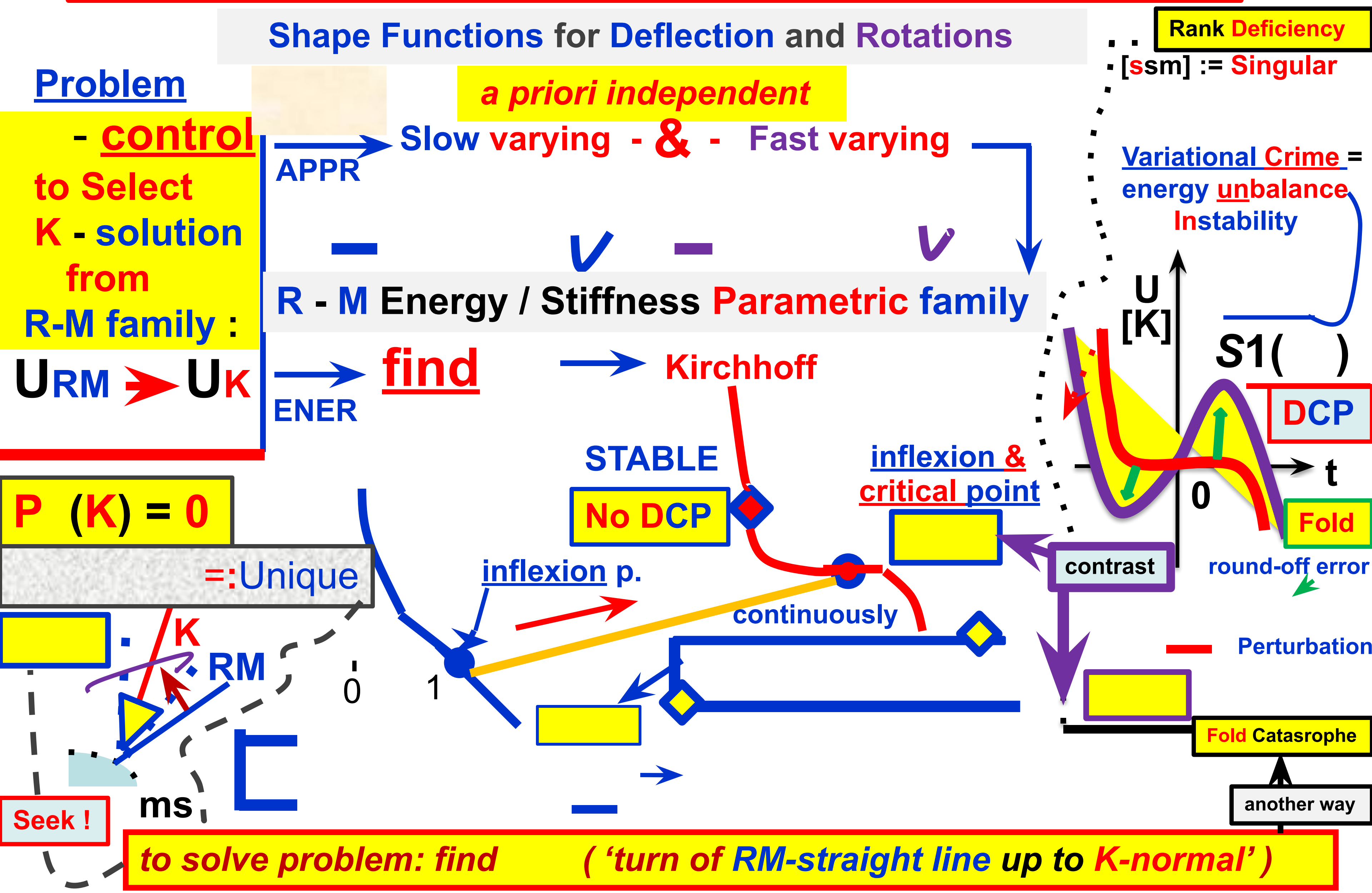
Catasrophe / Singularity Theory : Fold Catasrophe → Reaction to Small Perturbation

What the Energy (displacement) approximating level is TRUE for THIN plates ?

Unique Choice of : = ? → **plate mechanics** →

ENERGY Consistency of Field functions via APPROXIMATIONS

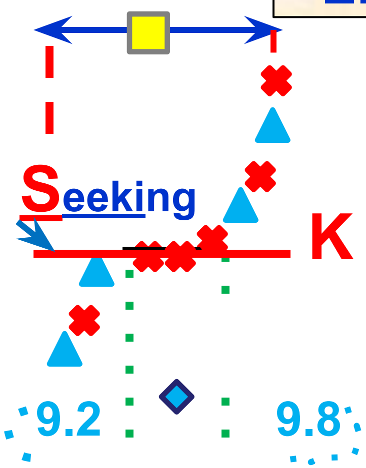
Kirchhoff case is a member from the Reissner – Mindlin family



Uniqueness of Critical Point of Inflexion: – finding

Selection of K – solution from Reissner – Mindlin family

' Energy via Deflection ' & FEM analysis data



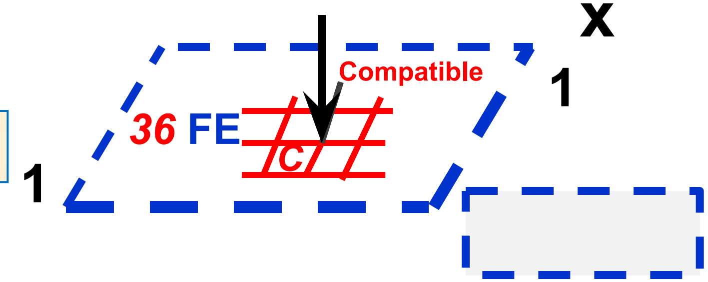
=

at Center

Cubic interpolation of FEM – data

interpolating points

control points FEM



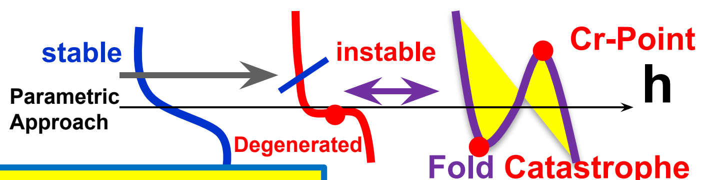
8 – node Kirchhoff – Reissner – Mindlin thin Plate FE

Consistency

P (K)=0

= [9.55 ; 9.75]

Consistency via MultiScale



Rank Deficiency

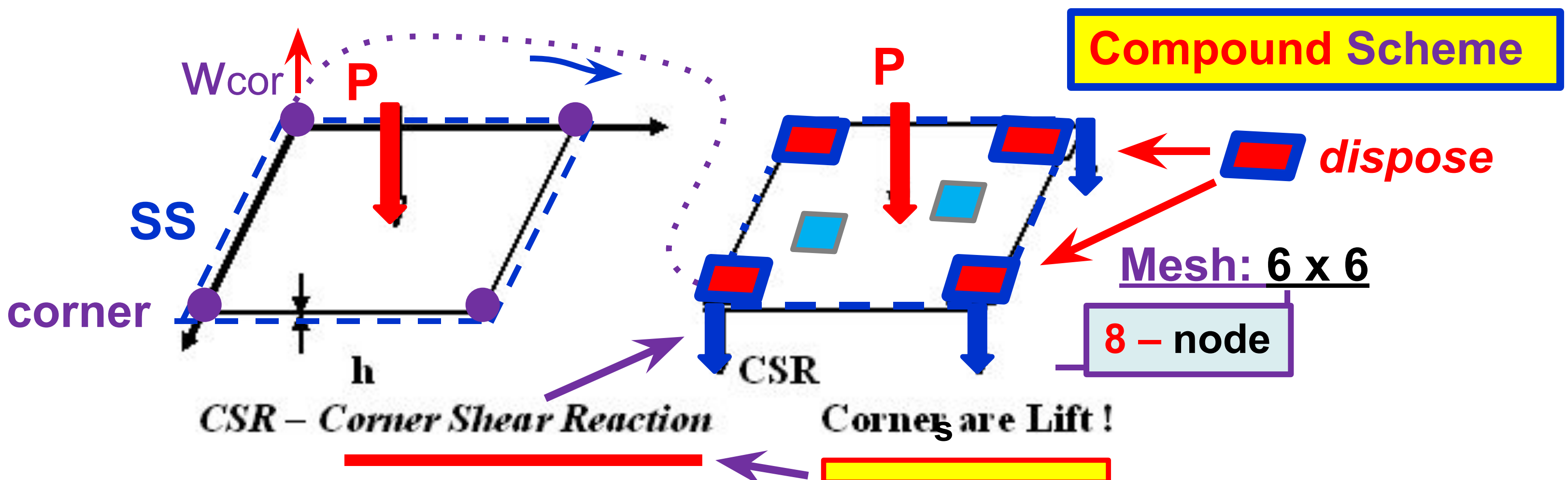
Reduced Integration

Round – off Error

FEM Stiff Problem of Solid Mechanics : *Reissner-Mindlin* Thin Plate Bending – *Shear Locking* Problem *& ROBUSTNESS*

$\{w, \theta^i\}$: **Displacement – based FEM**

Square Plate : a x a x h Simply Supported (SS - soft), loaded at the Center by a concentrated force P



Scheme:
- FI - Full Integration — 36 FE
- SR – Selective Reduced Integration — 36 FE
- C - Compound : (32 FE–SR) + (4 FE–KRM / Corners)

R – *Control via* Shape Functions OR *via* Variational Principle

8 - node

KRM: Constructed Kirchhoff-Reissner-Mindlin FE with *agreed C0 – deflection and rotations*

Convergence Improving (Quality Control)

Nondimensional Deflection at with varying (a/h) ratios

same mesh : 6 x 6

Large parameter (Stiff Problem):

relative error %

robust = + / -

stable = + / -

- ▪ +

\+ ▪ -

\+ ▪ +

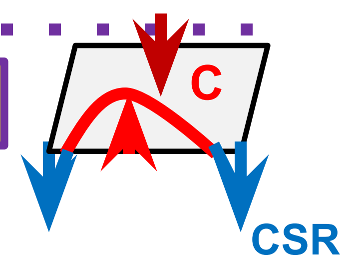
496

2

0.03

K-RM solution: Sch.

36 FE

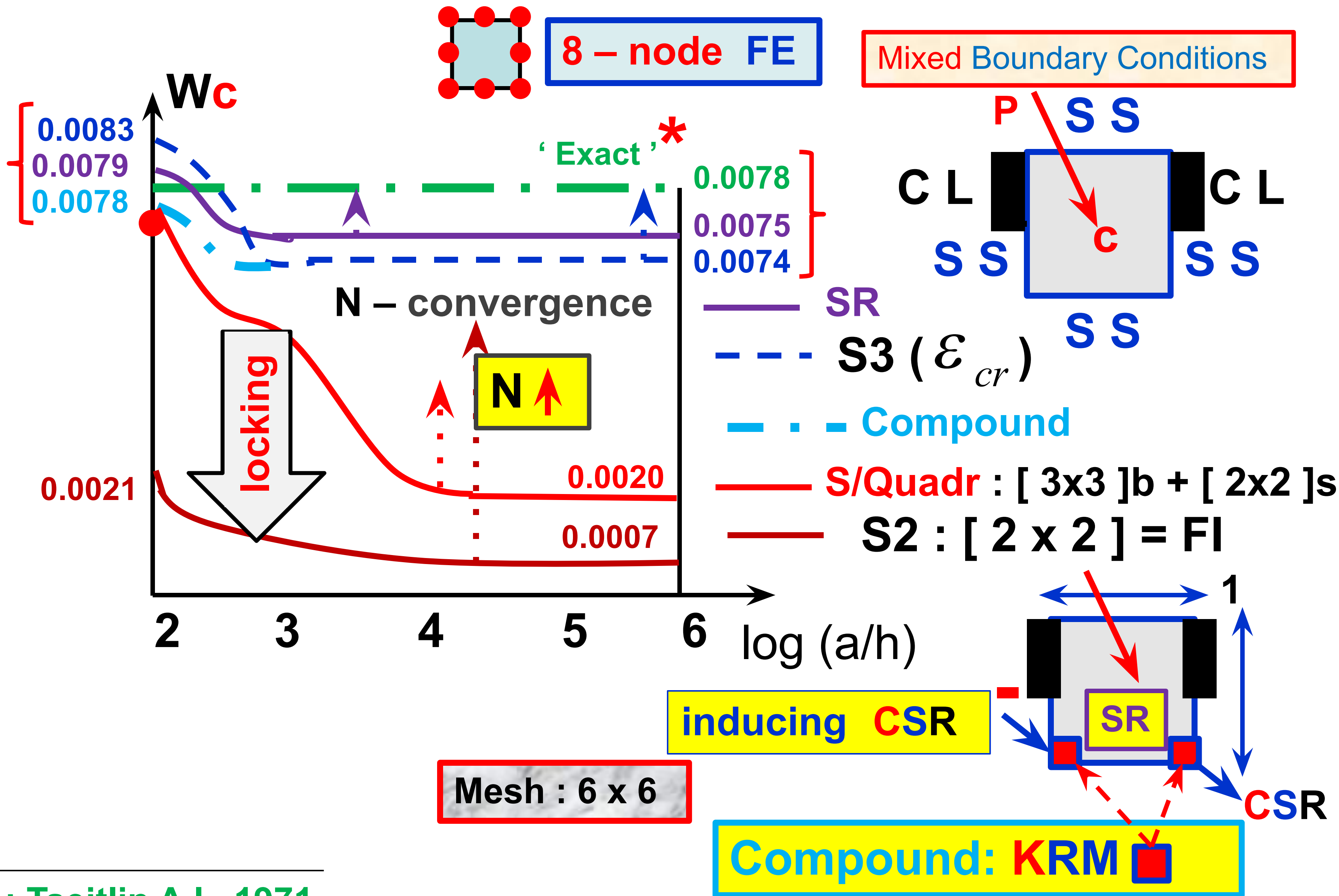


Quality = (Accuracy$_N$ + Stability) + Robustness

Method Stability : to Zero Energy Modes = Mechanisms

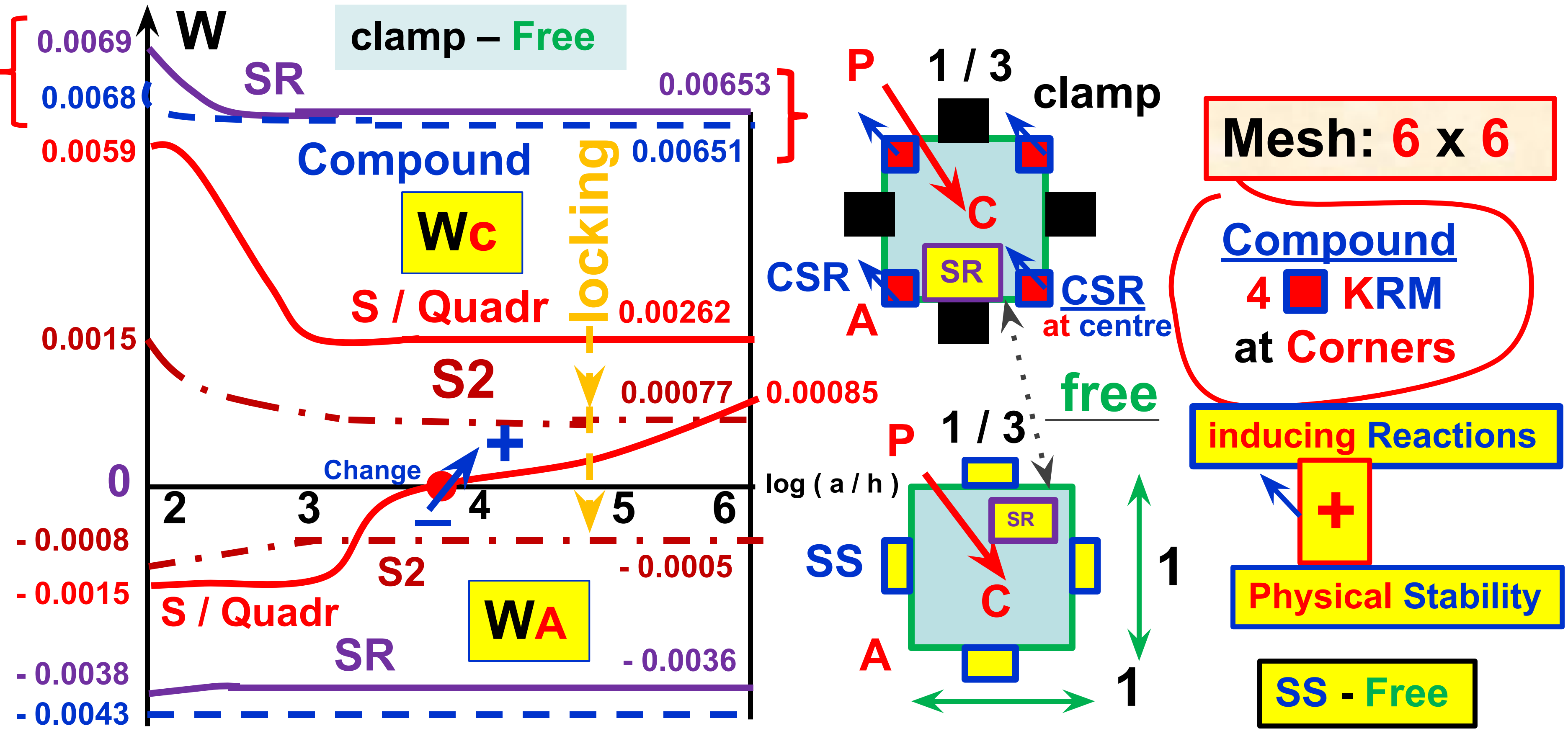
Robustness = Stability towards: Round-off error & Problem parameters

Thin plates with Strongly Connected Boundaries



* Ref. : Tseitlin A.I., 1971.

Thin plates with Strongly – Weakly Connected Boundaries



Compound

lg(a/h)	2	3	4	5	6	
W c	0.0134	0.0134	0.0134	0.0134	0.0134	0.0155
- WA	0.0048	0.0047	0.0047	0.0047	0.0047	0.0047

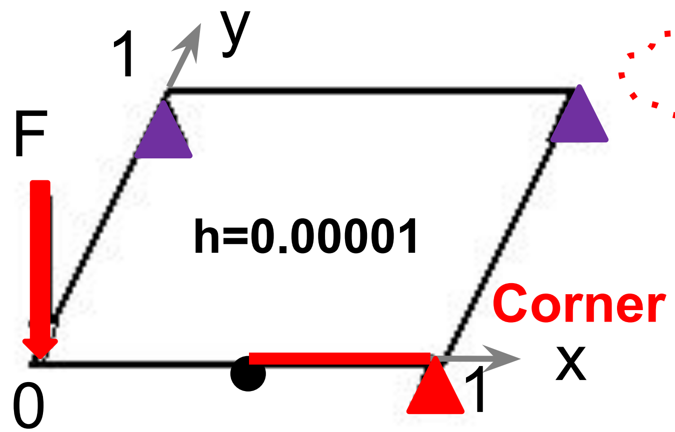
Ref. : Jiang Z., 1992

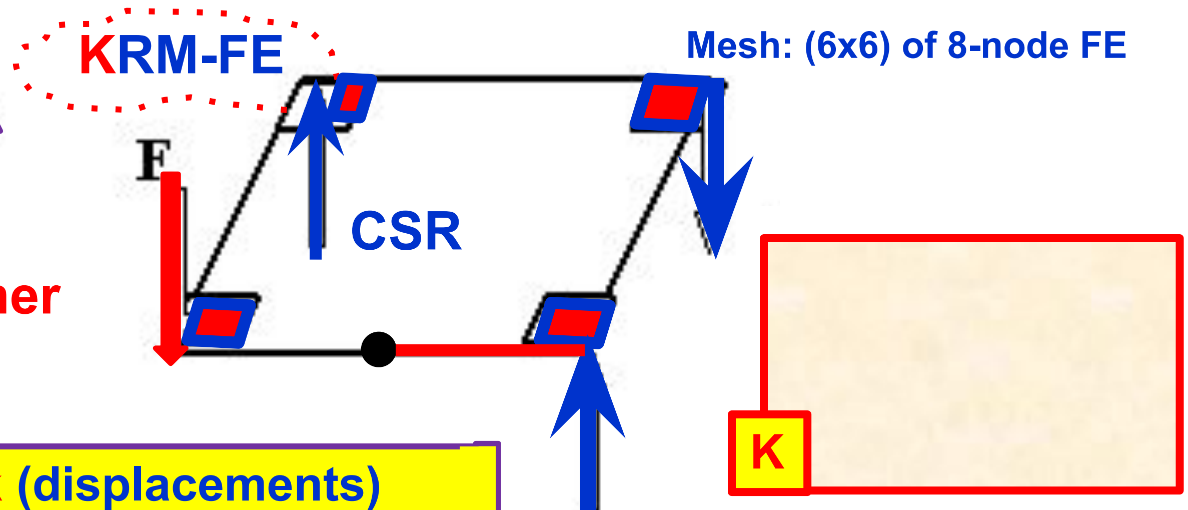
SS → 0

Point Singular Support

Reissner-Mindlin Thin Plate Bending – the case of Weak Connected boundaries / Zero Energy Modes

Torsion of Thin plate : 3 Node – Supported plate, loaded at the Corner by a concentrated force F





the principle of virtual work (displacements)

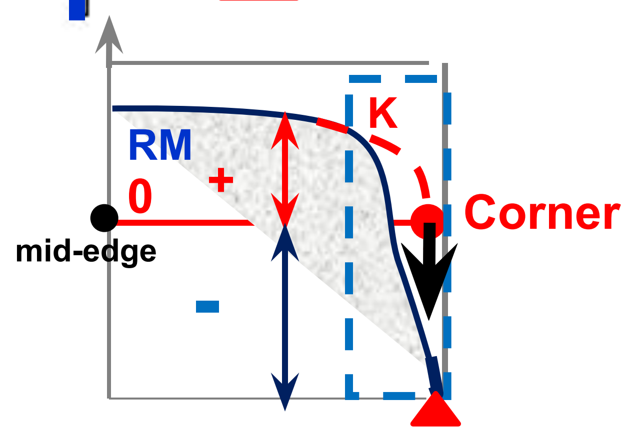
***variational* boundary conditions (*SS-soft*)**

RM:

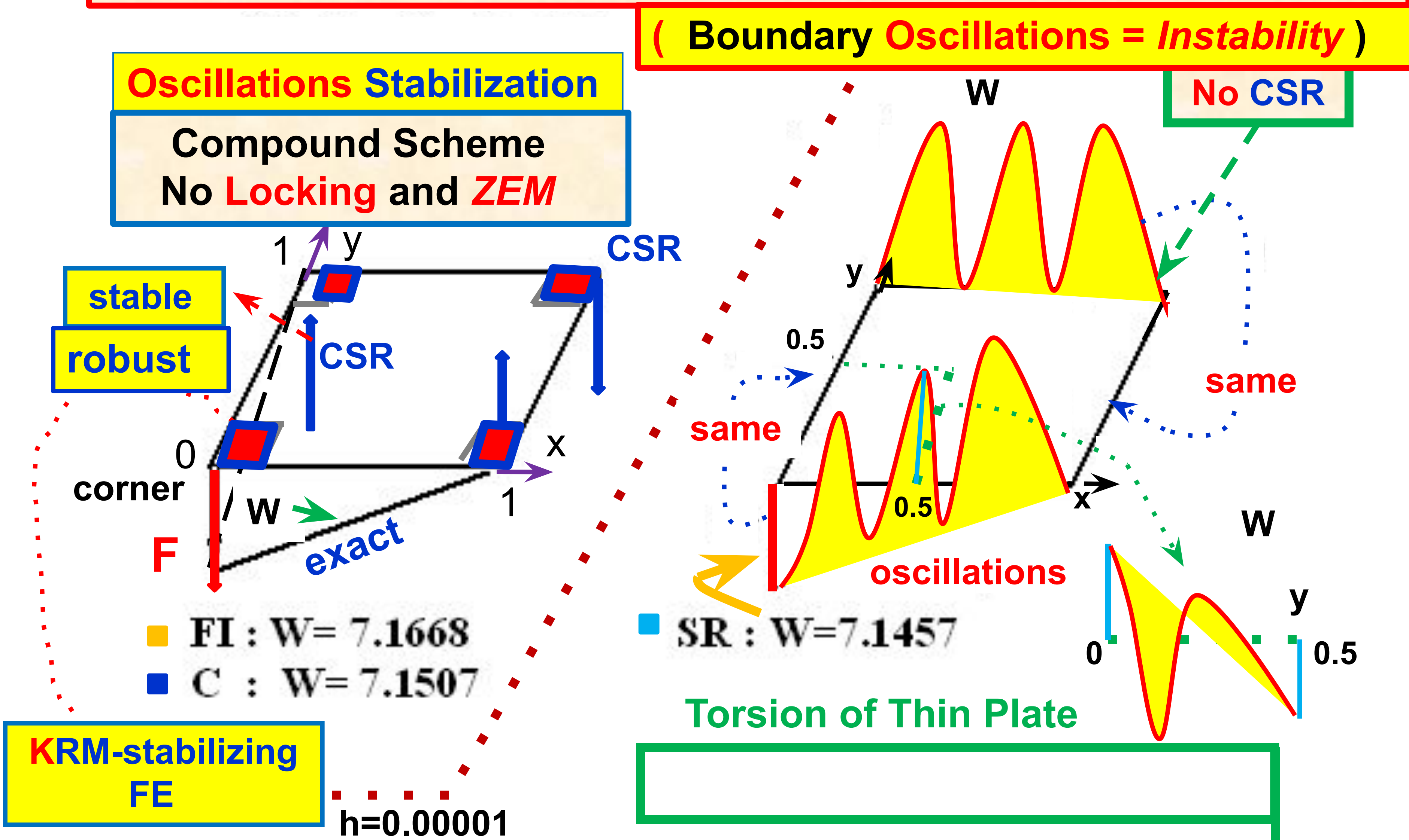
K:

essential

natural



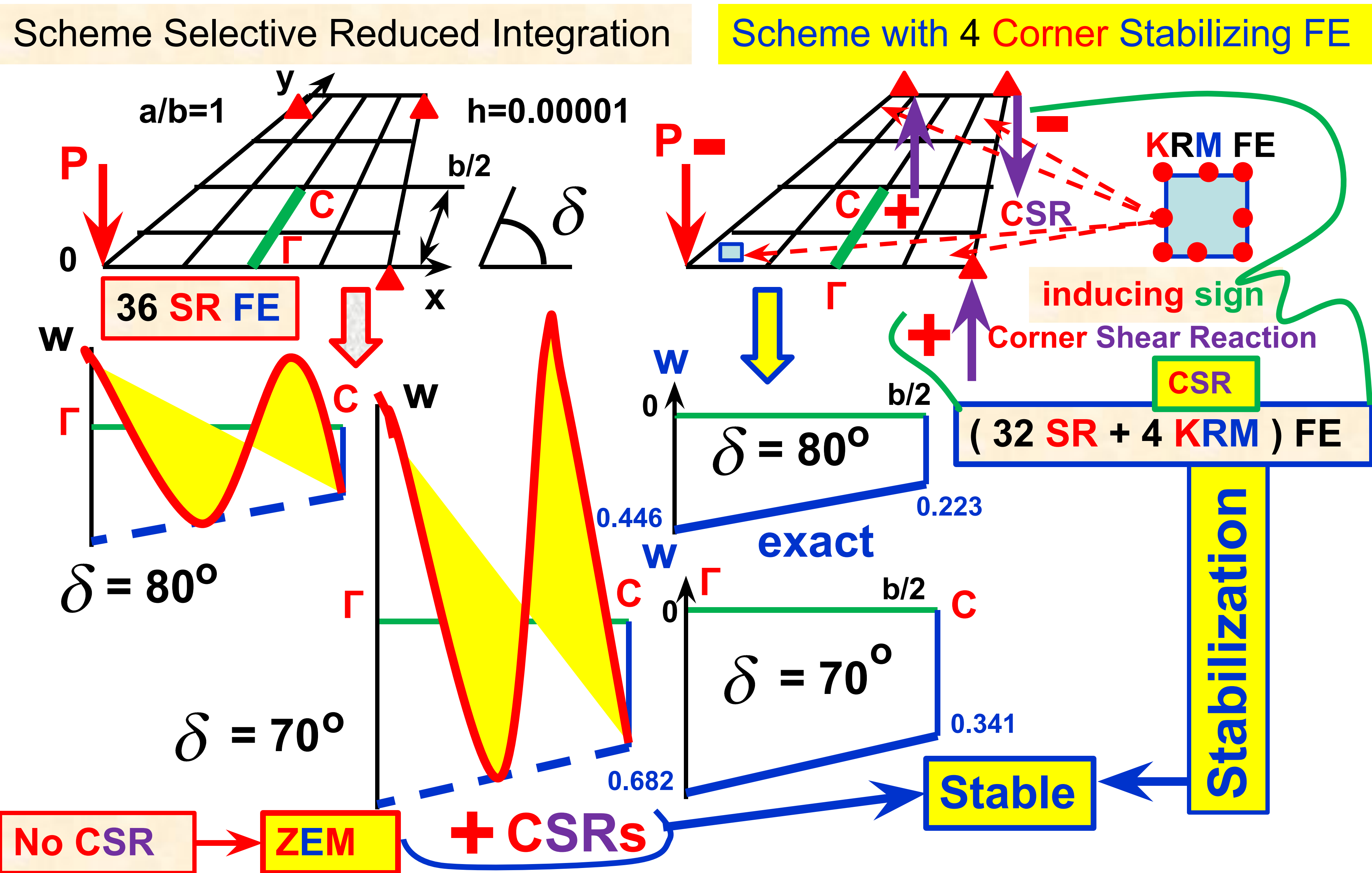
Selective Reduced Integration : *Zero Energy Modes*



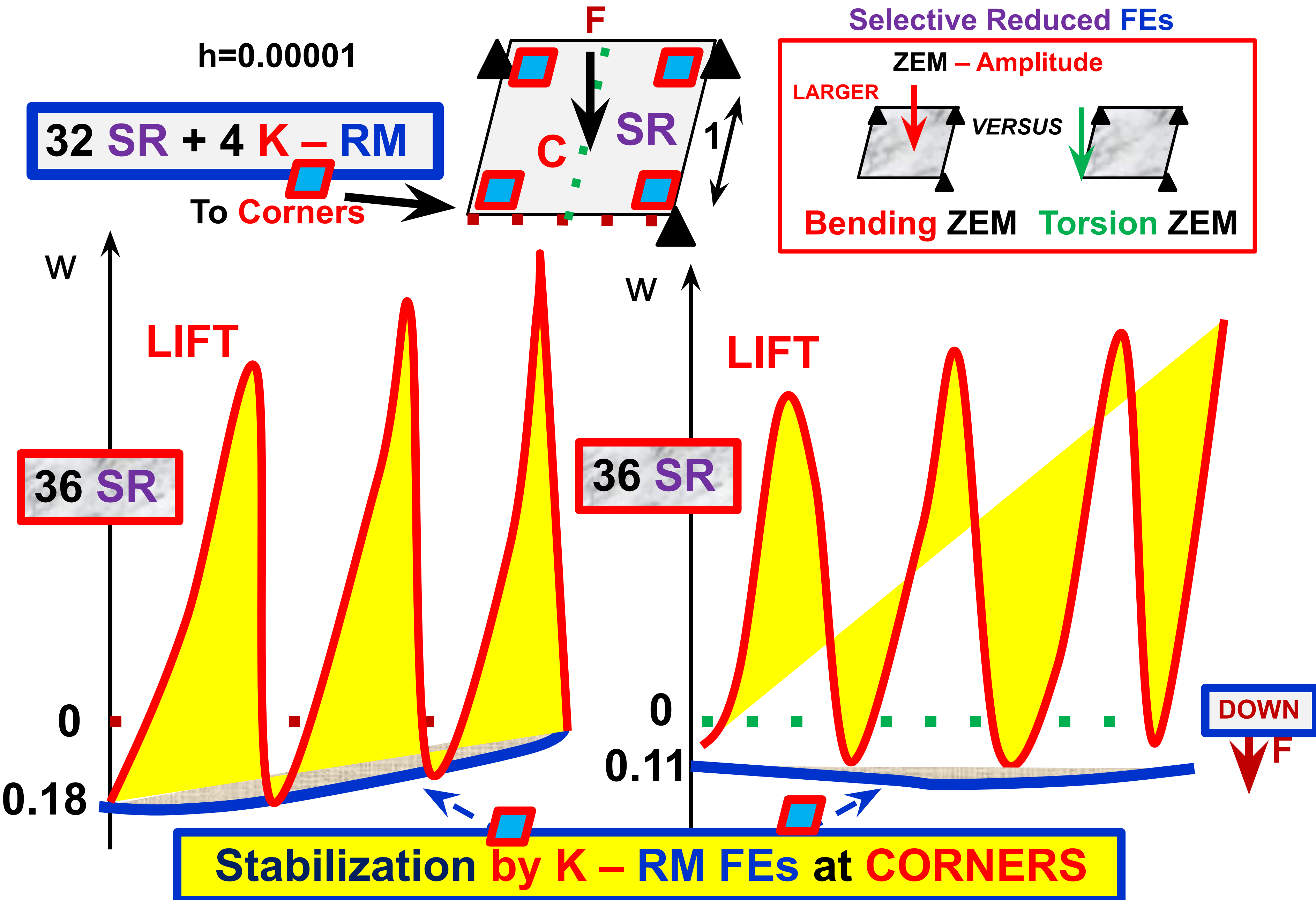
10 x Kirchhoff exact corner deflection = 7.1429 = W

Instability / Zero Energy Modes & Control by Stabilization

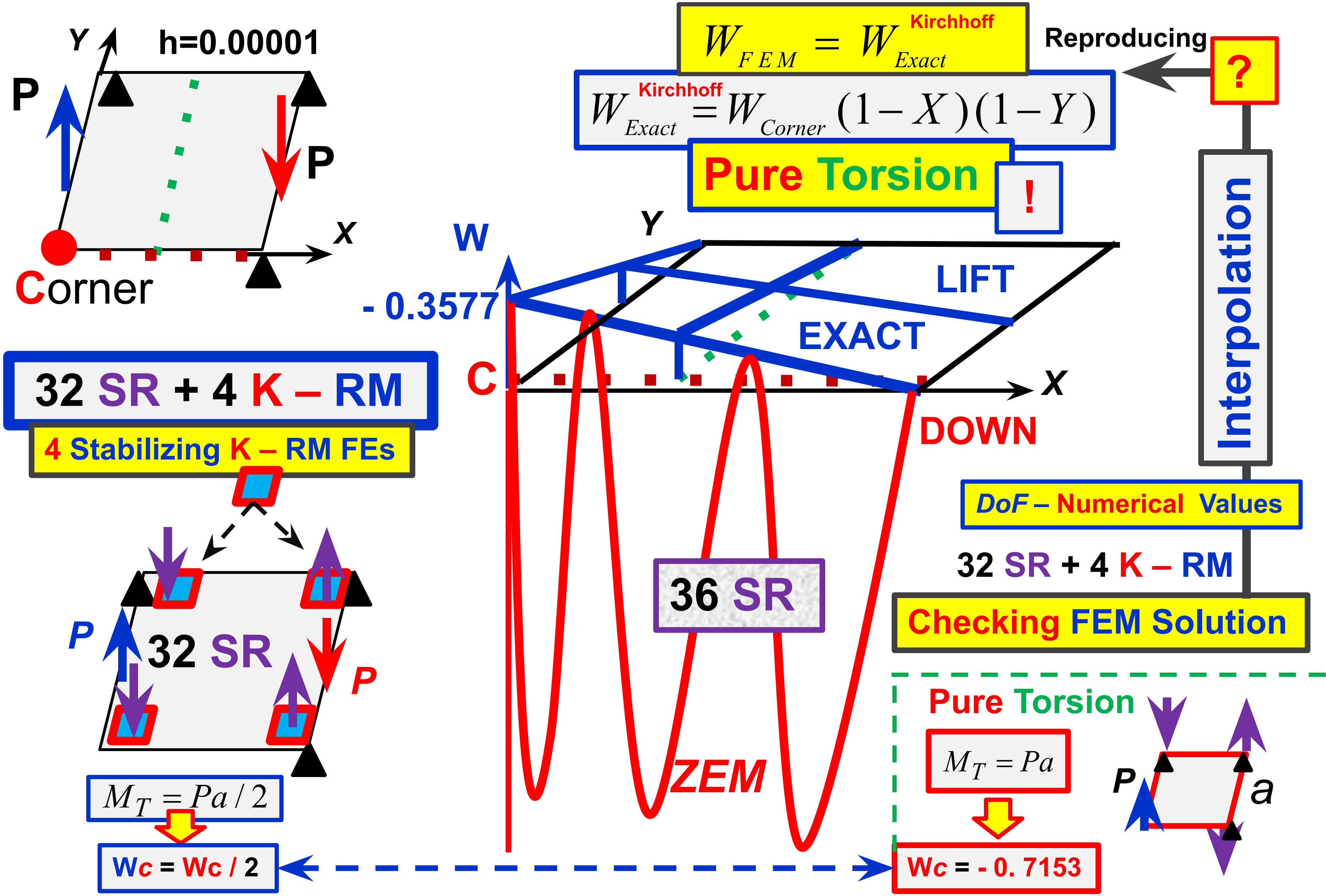
Trapezoidal Thin Plate : 3 Node – Supported – Torsion☆



3 Point Plate loaded at Center: increasing ZEM & Stabilization



Reissner – Mindlin Plate Bending: Identification of Torsion



4 – Point Singular Thin Plate Bending & Stabilization by RM Shear FEs

36 Selective Reduced

WILD Oscillations

Y=0. **LARGE ZEM**

X= .000	UZ=1350.9
X= .083	UZ= 675.4
X= .167	UZ= -.00892
X= .250	UZ= 675.4
X= .333	UZ=1350.9
X= .417	UZ= 675.4
X= .500	UZ= .0000
X= .583	UZ= 675.4
X= .667	UZ=1350.9
X= .750	UZ= 675.4
X= .833	UZ= .00218
X= .917	UZ= 675.4
X= 1.000	UZ=1350.9

X=0.5

Y= .000	UZ= .0000
Y= .083	UZ= 675.4
Y= .167	UZ=1350.9
Y= .250	UZ= 675.4
Y= .333	UZ= -.00197
Y= .417	UZ= 675.4
Y= .500	UZ=1350.9
Y= .583	UZ= 675.4
Y= .667	UZ= -.000018
Y= .750	UZ= 675.4
Y= .833	UZ=1350.9
Y= .917	UZ= 675.4
Y= 1.000	UZ= .0000

Oscillations

32 SR + 4 RM Shear

No Oscillations

Jiang & Liu, exact

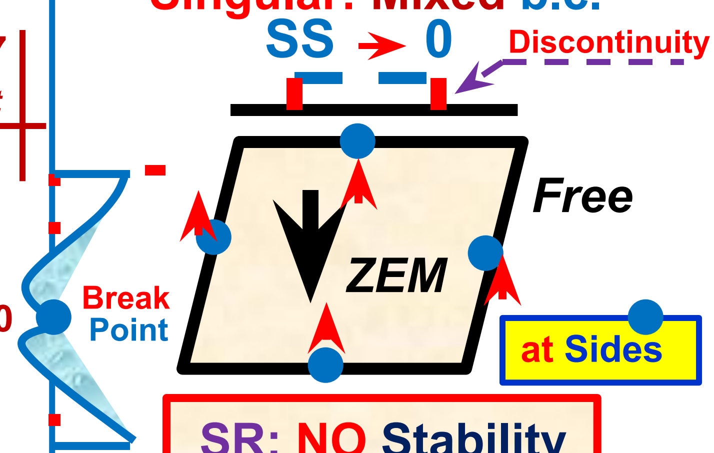
UZ= -.004015	UKT= -.004727
UZ= -.002307	
UZ= -.001067	
UZ= -.000080	
UZ= .000928	
UZ= .000453	
UZ= .000000	UKT= .000000
UZ= .000453	
UZ= .000928	
UZ= -.000080	
UZ= -.001067	
UZ= -.002307	
UZ= -.004015	UKT= -.004727

X=0.5

UZ= .000000	UKT= .000000
UZ= .003395	
UZ= .006811	
UZ= .008794	
UZ= .010798	
UZ= .012394	
UZ= .014011	UKT= .015456
UZ= .012394	
UZ= .010798	
UZ= .008794	
UZ= .006811	
UZ= .003395	
UZ= .000000	UKT= .000000

h = 0.00001 ; Mesh : 6 x 6

Singular: Mixed b.c.



SR: NO Stability

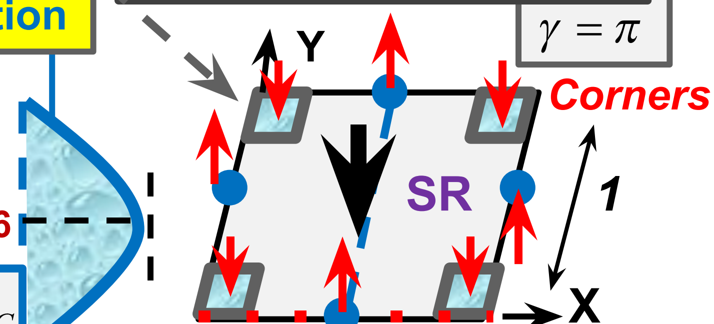
Rank Deficiency *Crime* $[K_s] = \sin g$

32 SR + 4 RM Shear FEs

$\{w(\varepsilon_0 = \gamma), \theta^i(\varepsilon), \varepsilon = \gamma\}$

$\gamma = \pi$

Stabilization



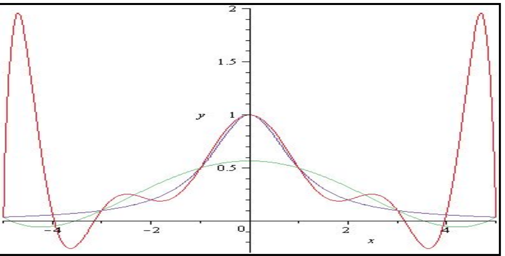
$U = 1/2P(w)_C$

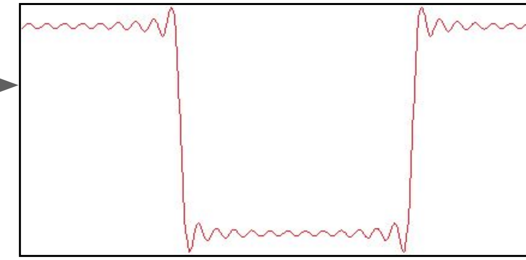
$\delta\Pi = 0 \rightarrow \min\Pi \rightarrow \{U_i\}$

State of Equilibrium

Appendix: Spectral Non – Algebraic Shape Functions Properties

1D, 2D, 3D Interpolations for Uniformly Spaced Nodes





For Optimal Nodes **NO** Runge Phenomenon.

For Complete Interpolation Bases **NO** Gibbs Phenomenon.

For Arbitrary Number of Boundary Nodes **NO** Internal Nodes.

Expansions into the Shape Functions series