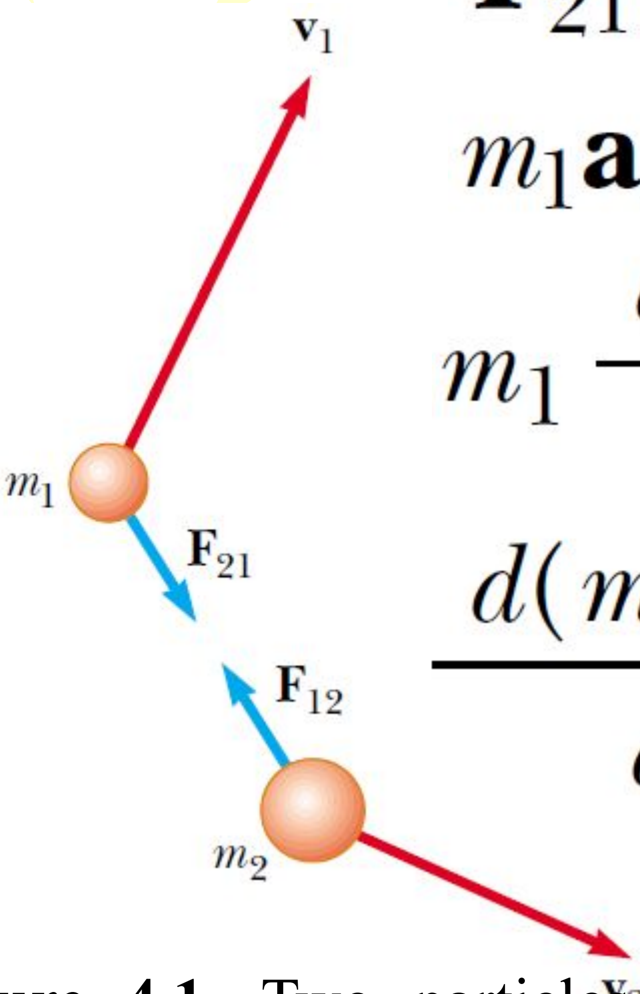
Course of lectures «Contemporary Physics: Part1»

Lecture №5

Linear Momentum and Collisions.

Rotation of a Rigid Object about a Fixed Axis.

Linear Momentum and Its Conservation



$$\mathbf{F}_{21} + \mathbf{F}_{12} = 0$$

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = 0$$

$$m_1 \frac{d\mathbf{v}_1}{dt} + m_2 \frac{d\mathbf{v}_2}{dt} = 0$$

$$\frac{d(m_1\mathbf{v}_1)}{dt} + \frac{d(m_2\mathbf{v}_2)}{dt} = 0$$

Figure 4.1. Two particles interact with each other. According to Newton's third law, we must have $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

$$\frac{d}{dt}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2) = 0 \qquad \textbf{(4.1)}$$

The **linear momentum** of a particle or an object that can be modeled as a particle of mass m moving with a velocity **v** is defined to be the product of the mass and velocity:

$$\mathbf{p} \equiv m\mathbf{v} \qquad \textbf{(4.2)}$$

Linear momentum is a vector quantity because it equals the product of a scalar quantity m and a vector quantity **v**. Its direction is along **v**, it has dimensions ML/T, and its SI unit is kg · m/s.

If a particle is moving in an arbitrary direction, **p** must have three components

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z$$

As you can see from its definition, the concept of momentum provides a quantitative distinction between heavy and light particles moving at the same velocity. For example, the momentum of a bowling ball moving at 10 m/s is much greater than that of a tennis ball moving at the same speed. Newton called the product m**v** *quantity of motion*; this is perhaps a more graphic description than our present-day word *momentum*, which comes from the Latin word for movement.

Using Newton's second law of motion, we can relate the linear momentum of a particle to the resultant force acting on the particle. We start with Newton's second law and substitute the definition of acceleration:

$$\sum \mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$$

As m=const:

$$\sum \mathbf{F} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt} \qquad \textbf{(4.3)}$$

The time rate of change of the linear momentum of a particle is equal to the net force acting on the particle.

Using the definition of momentum, Equation 4.1 can be written

$$\frac{d}{dt}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2) = 0$$

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

$$\mathbf{p}_{\text{tot}} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant} \qquad \textbf{(4.4)}$$

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \qquad \textbf{(4.5)}$$

where $\mathbf{p}_{1i}$ and $\mathbf{p}_{2i}$ are the initial values and $\mathbf{p}_{1f}$ and $\mathbf{p}_{2f}$ the final values of the momenta for the two particles for the time interval during which the particles interact.

$$p_{ix} = p_{fx} \qquad p_{iy} = p_{fy} \qquad p_{iz} = p_{fz} \qquad \textbf{(4.6)}$$

This result, known as the **law of conservation of linear momentum**, can be extended to any number of particles in an isolated system. It is considered one of the most important laws of mechanics. We can state it as follows:

Whenever two or more particles in an isolated system interact, the total momentum of the system remains constant.

This law tells us that the total momentum of an isolated system at all times equals its initial momentum.

Notice that we have made no statement concerning the nature of the forces acting on the particles of the system. The only requirement is that the forces must be *internal* to the system.

Impulse and Momentum

The momentum of a particle changes if a net force acts on the particle.

$$\sum \mathbf{F} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}$$

According to Newton's second law

$$d\mathbf{p} = \mathbf{F}\,dt \qquad \textbf{(4.7)}$$

$$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F}\,dt \qquad \textbf{(4.8)}$$

To evaluate the integral, we need to know how the force varies with time. The quantity on the right side of this equation is called the **impulse** of the force **F** acting on a particle over the time interval $\Delta t = t_f - t_i$. Impulse is a vector defined by

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F}\, dt \qquad \textbf{(4.9)}$$

Equation 4.8 is an important statement known as the impulse–momentum theorem:

$$\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F}\, dt$$

The impulse of the force **F** acting on a particle equals the change in the momentum of the particle.

The direction of the impulse vector is the same as the direction of the change in momentum. Impulse has the dimensions of momentum—that is, ML/T. Note that impulse is ***not*** a property of a particle; rather, it is a measure of the degree to which an external force changes the momentum of the particle. Therefore, when we say that an impulse is given to a particle, we mean that momentum is transferred from an external agent to that particle.

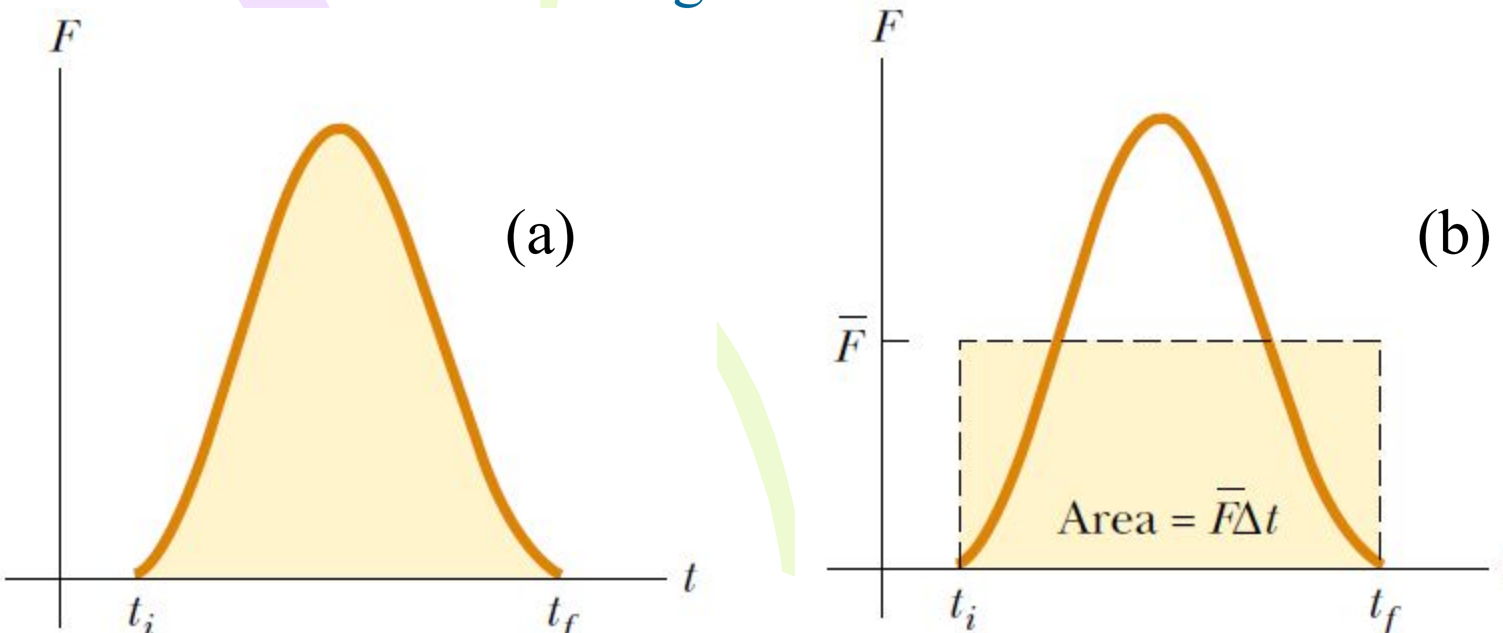


Figure 4.2 (a) A force acting on a particle may vary in time. The impulse imparted to the particle by the force is the area under the force-versus-time curve. (b) In the time interval (t, *the time-averaged* force (horizontal dashed line) gives the same impulse to a particle as does the time-varying force described in part (a).

Because the force imparting an impulse can generally vary in time, it is convenient to define a time-averaged force

$$\overline{\mathbf{F}} \equiv \frac{1}{\Delta t}\int_{t_i}^{t_f} \mathbf{F}\, dt \qquad \textbf{(4.10)}$$

where $\Delta t = t_f - t_i$.

$$\mathbf{I} \equiv \overline{\mathbf{F}}\,\Delta t \qquad \textbf{(4.11)}$$

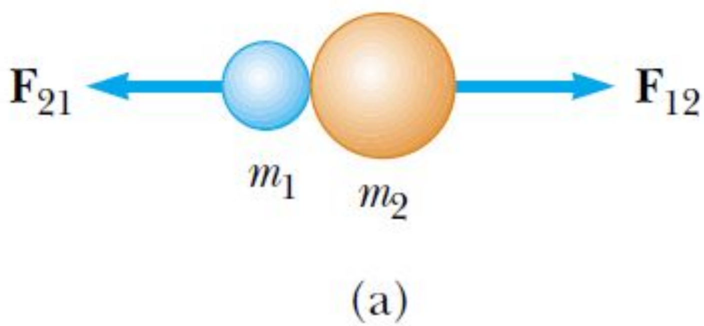
The calculation becomes especially simple if the force acting on the particle is constant. In this case, $\overline{\mathbf{F}} = \mathbf{F}$ and Equation 4.11 becomes

$$\mathbf{I} = \mathbf{F}\,\Delta t \qquad \textbf{(4.12)}$$

In many physical situations, we shall use what is called the **impulse approximation, in which we assume that one of the forces exerted on a particle acts for a short time but is much greater than any other force present**.

Collisions in One Dimension

We use the term collision to represent an event during which two particles come close to each other and interact by means of forces. The time interval during which the velocities of the particles change from initial to final values is assumed to be short. The interaction forces are assumed to be much greater than any external forces present, so we can use the impulse approximation.



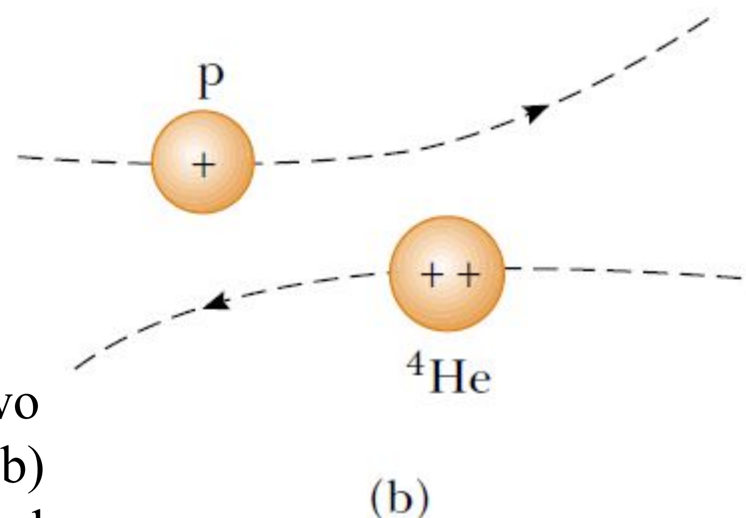


Figure 4.3 (a) The collision between two objects as the result of direct contact. (b) The "collision" between two charged particles.

The total momentum of an isolated system just before a collision equals the total momentum of the system just after the collision.

The total kinetic energy of the system of particles may or may not be conserved, depending on the type of collision. In fact, whether or not kinetic energy is conserved is used to classify collisions as either *elastic* or *inelastic*.

An **elastic collision** between two objects is one in which the **total kinetic energy (as well as total momentum) of the system is the same before and after the collision**. Collisions between certain objects in the macroscopic world, such as billiard balls, are only *approximately* elastic because some deformation and loss of kinetic energy take place. For example, you can hear a billiard ball collision, so you know that some of the energy is being transferred away from the system by sound. An elastic collision must be perfectly silent! *Truly* elastic collisions occur between atomic and subatomic particles.

An **inelastic collision** is one in which **the total kinetic energy of the system is not the same before and after the collision (even though the momentum of the system is conserved)**.

Inelastic collisions are of two types.

When the colliding objects stick together after the collision, as happens when a meteorite collides with the Earth, the collision is called **perfectly inelastic**.

When the colliding objects do not stick together, but some kinetic energy is lost, as in the case of a rubber ball colliding with a hard surface, the collision is called **inelastic** (with no modifying adverb).

In most collisions, the kinetic energy of the system is *not* conserved because some of the energy is converted to internal energy and some of it is transferred away by means of sound. Elastic and perfectly inelastic collisions are limiting cases; most collisions fall somewhere between them.

The important distinction between these two types of collisions is that **momentum of the system is conserved in all collisions, but kinetic energy of the system is conserved only in elastic collisions.**

Perfectly Inelastic Collisions

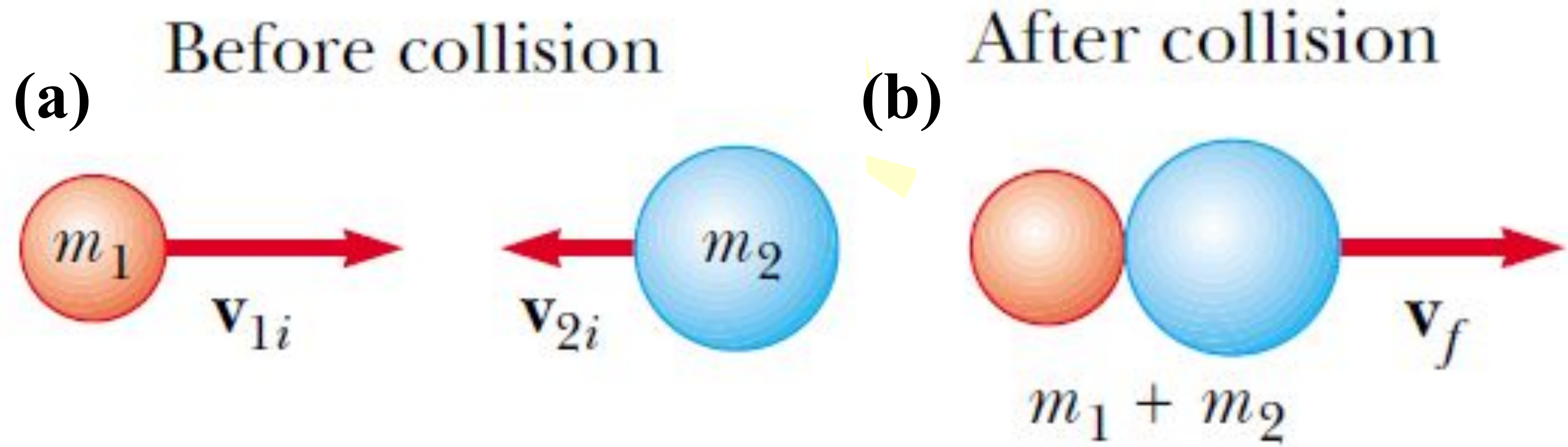


Figure 4.4 Schematic representation of a perfectly inelastic head-on collision between two particles: (a) before collision and (b) after collision.

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f \quad \textbf{(4.13)}$$

$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2} \quad \textbf{(4.14)}$$

Elastic Collisions

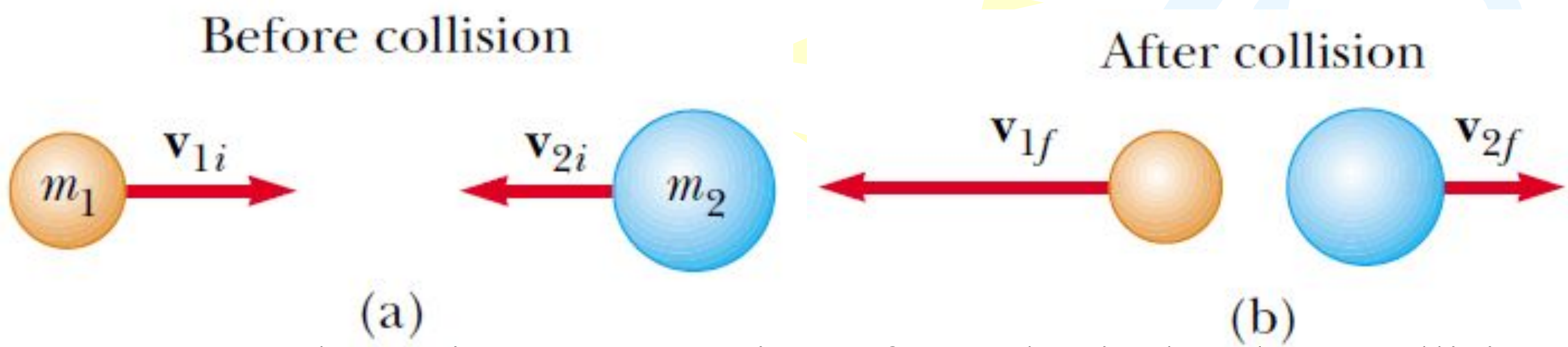


Figure 4.5 Schematic representation of an elastic head-on collision between two particles: (a) before collision and (b) after collision.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad \textbf{(4.15)}$$

$$\frac{1}{2} m_1 {v_{1i}}^2 + \frac{1}{2} m_2 {v_{2i}}^2 = \frac{1}{2} m_1 {v_{1f}}^2 + \frac{1}{2} m_2 {v_{2f}}^2 \quad \textbf{(4.16)}$$

$$m_1({v_{1i}}^2 - {v_{1f}}^2) = m_2({v_{2f}}^2 - {v_{2i}}^2)$$

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i})$$

(4.17)

Next, let us separate the terms containing m_1 and m_2 in Equation 4.15 to obtain

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \quad \textbf{(4.18)}$$

To obtain our final result, we divide Equation 4.17 by Equation 4.18 and obtain

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \quad \textbf{(4.19)}$$

Suppose that the masses and initial velocities of both particles are known.

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i} \qquad \textbf{(4.20)}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i} \qquad \textbf{(4.21)}$$

Let us consider some special cases. If $m_1 = m_2$, then Equations 4.20 and 4.21 show us that $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$.

That is, the particles exchange velocities if they have equal masses. This is approximately what one observes in head-on billiard ball collisions - the cue ball stops, and the struck ball moves away from the collision with the same velocity that the cue ball had.

If particle 2 is initially at rest, then $v_{2i} = 0$, and Equations 4.20 and 4.21 become

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \qquad \textbf{(4.22)}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \qquad \textbf{(4.23)}$$

If m_1 is much greater than m_2 and $v_{2i} = 0$, we see from Equations 4.22 and 4.23 that $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$.

If m_2 is much greater than m_1 and particle 2 is initially at rest, then $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx 0$.

Two-Dimensional Collisions

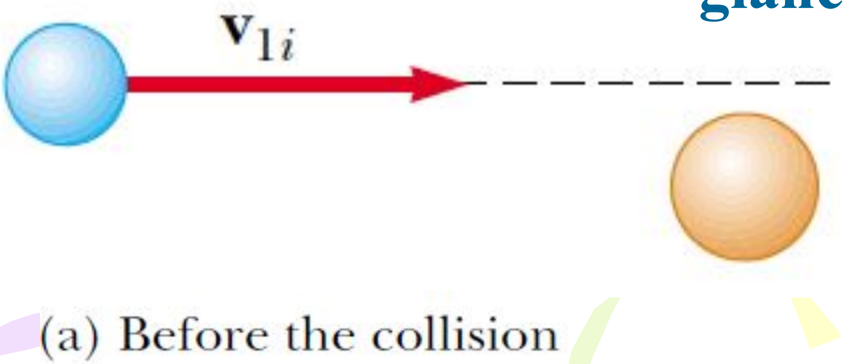
The momentum of a system of two particles is conserved when the system is isolated. For any collision of two particles, this result implies that the momentum in each of the directions x, y, and z is conserved.

For such two-dimensional collisions, we obtain two component equations for conservation of momentum:

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

glancing collision



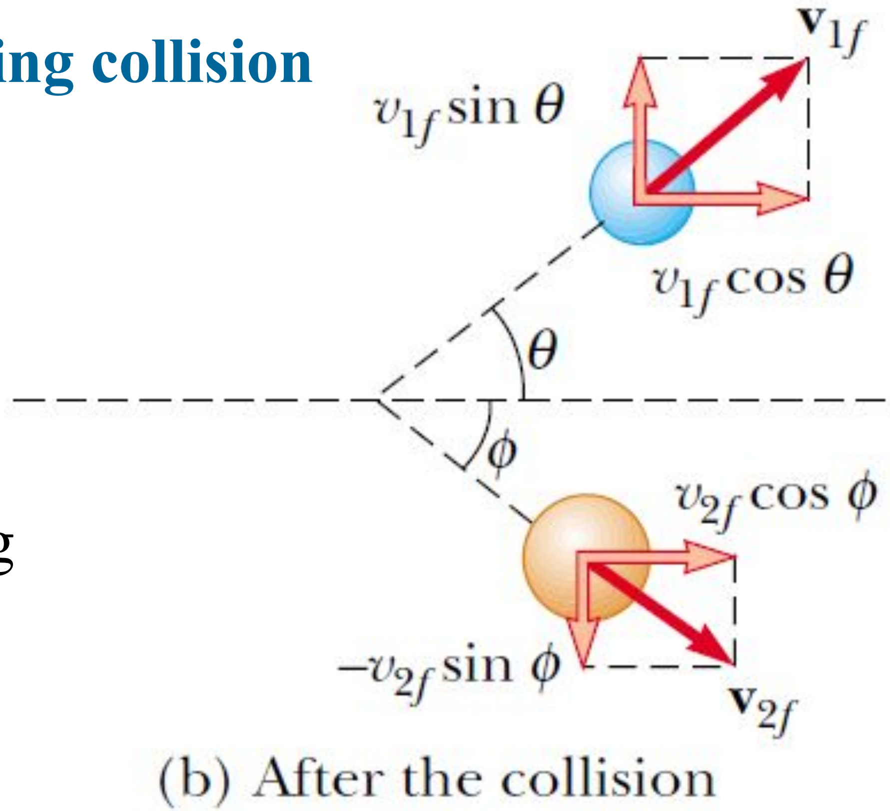


Figure 4.6 An elastic glancing collision between two particles.

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi \quad \textbf{(4.24)}$$

$$0 = m_1 v_{1f} \sin\theta - m_2 v_{2f} \sin\phi \quad \textbf{(4.25)}$$

If the collision is elastic, we can also use Equation 4.16 (conservation of kinetic energy) with $v_{2i} = 0$ to give

$$\frac{1}{2}m_1 {v_1}_i^2 = \frac{1}{2}m_1 {v_1}_f^2 + \frac{1}{2}m_2 {v_2}_f^2 \qquad \textbf{(4.26)}$$

If the collision is inelastic, kinetic energy is not conserved and Equation 4.26 does *not* apply.

The Center of Mass

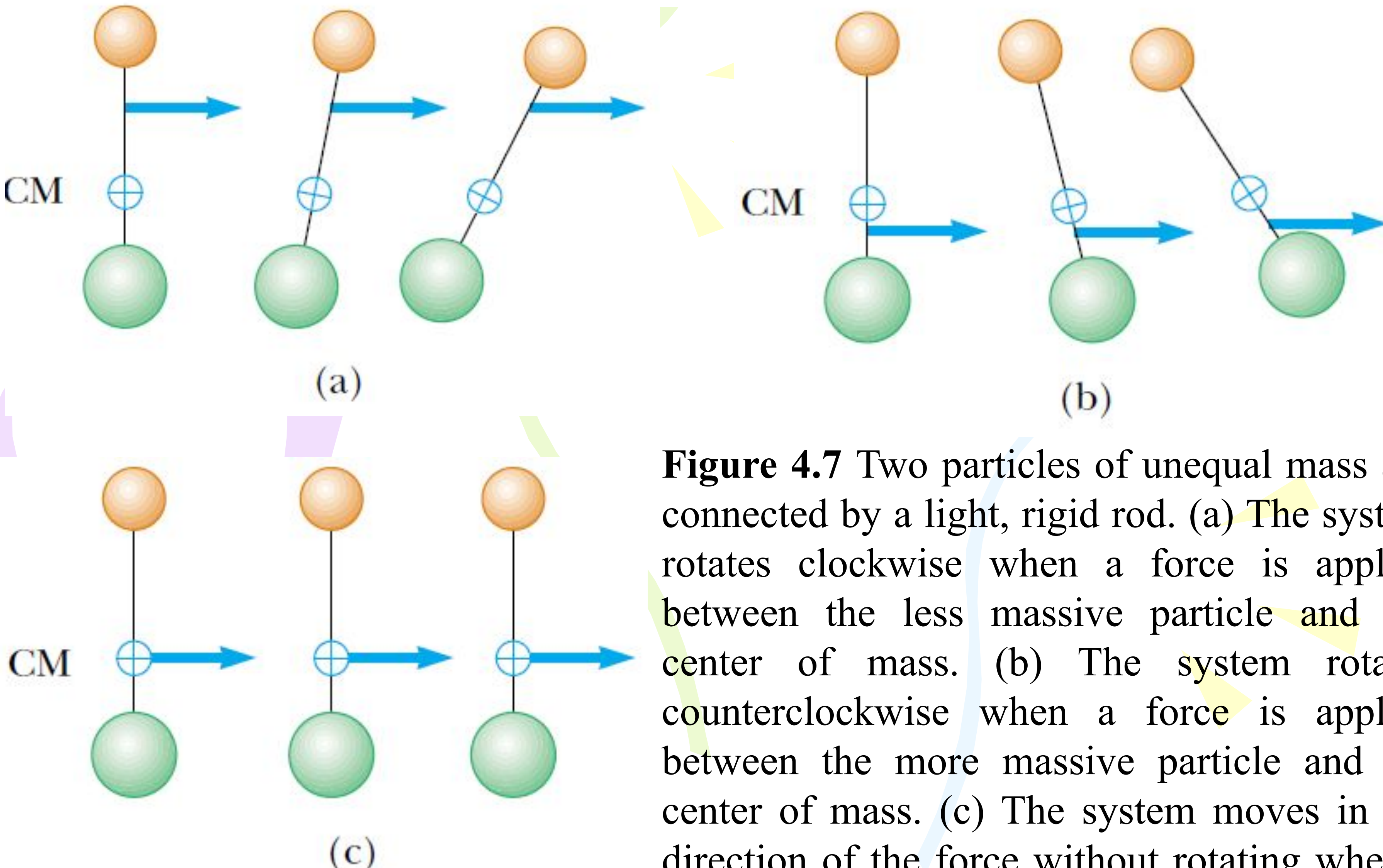
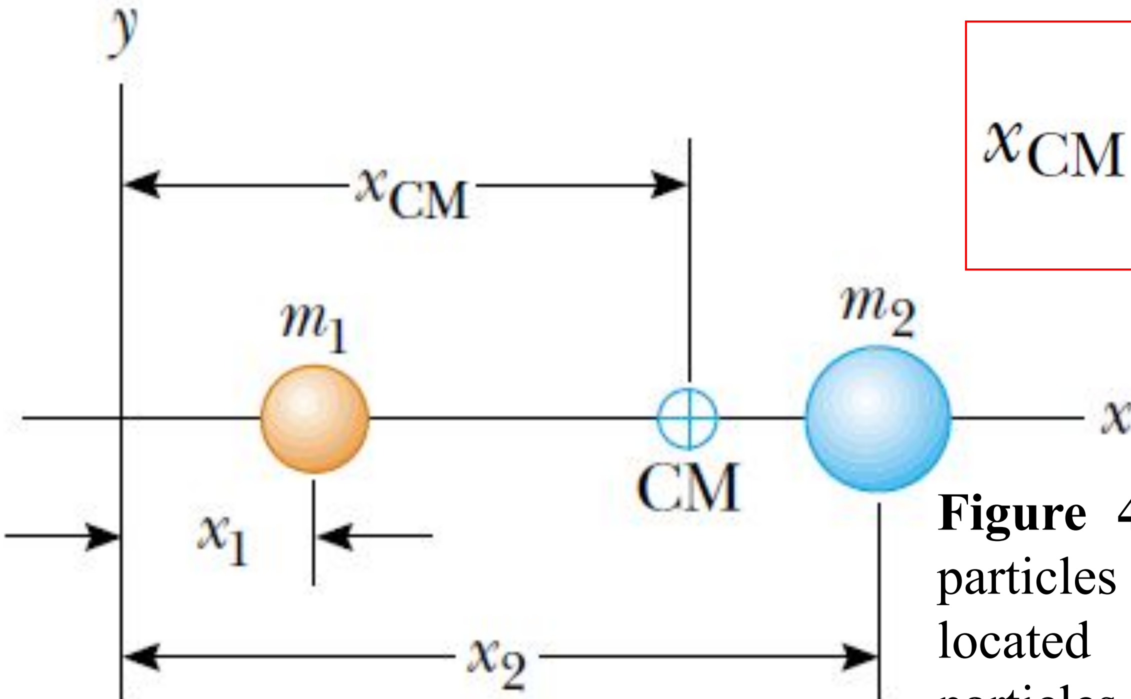


Figure 4.7 Two particles of unequal mass are connected by a light, rigid rod. (a) The system rotates clockwise when a force is applied between the less massive particle and the center of mass. (b) The system rotates counterclockwise when a force is applied between the more massive particle and the center of mass. (c) The system moves in the direction of the force without rotating when a force is applied at the center of mass.



$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

(4.27)

Figure 4.8 The center of mass of two particles of unequal mass on the x axis is located at x_{CM}, a point between the particles, closer to the one having the larger mass.

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{M}$$

(4.28)

$$y_{CM} \equiv \frac{\sum_i m_i y_i}{M} \quad \text{and} \quad z_{CM} \equiv \frac{\sum_i m_i z_i}{M}$$

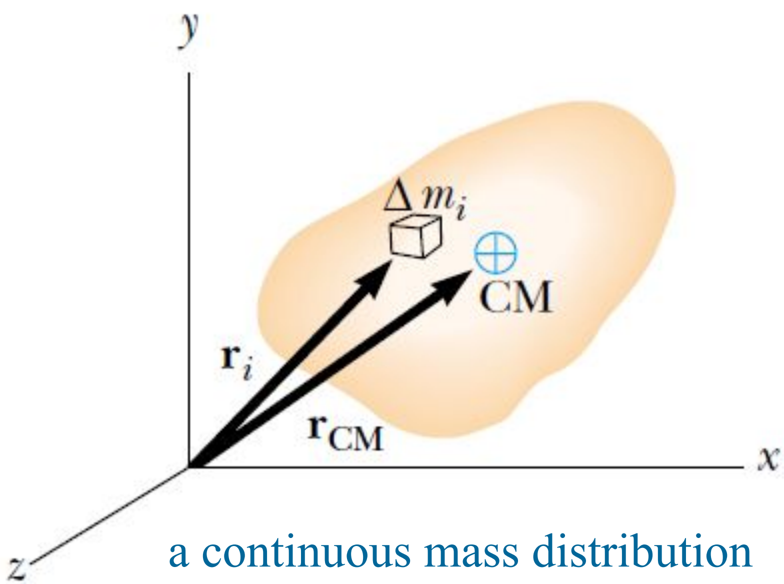
(4.29)

$$\mathbf{r}_{CM} \equiv \frac{\sum_i m_i \mathbf{r}_i}{M} \qquad (4.30)$$

$$\mathbf{r}_i \equiv x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

The center of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry.

$$\mathbf{r}_{CM} = \frac{1}{M}\int \mathbf{r}\, dm$$



a continuous mass distribution

Figure 4.9 An extended object can be considered to be a distribution of small elements of mass Δm_i. The center of mass is located at the vector position r_{CM}, which has coordinates x_{CM}, y_{CM} and z_{CM}

Motion of a System of Particles

Assuming M remains constant for a system of particles, that is, no articles enter or leave the system, we obtain the following expression for **the velocity of the center of mass** of the system:

$$\mathbf{v}_{\mathrm{CM}} = \frac{d\mathbf{r}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{r}_i}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{M} \qquad \textbf{(4.34)}$$

where v_i is the velocity of the i_{th} particle. Rearranging Equation 4.34 gives

$$M\mathbf{v}_{\mathrm{CM}} = \sum_i m_i \mathbf{v}_i = \sum_i \mathbf{p}_i = \mathbf{p}_{\mathrm{tot}} \qquad \textbf{(4.35)}$$

Therefore, we conclude that the **total linear momentum of the system equals the total mass multiplied by the velocity of the center of mass**. In other words, the total linear momentum of the system is equal to that of a single particle of mass *M* moving with a velocity $\mathbf{v}_{\mathbf{CM}}$.

If we now differentiate Equation 4.34 with respect to time, we obtain the **acceleration of the center of mass** of the system:

$$\mathbf{a}_{CM} = \frac{d\mathbf{v}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{v}_i}{dt} = \frac{1}{M}\sum_i m_i \mathbf{a}_i \quad \textbf{(4.36)}$$

Rearranging this expression and using Newton's second law, we obtain

$$M\mathbf{a}_{CM} = \sum m_i \mathbf{a}_i = \sum_i \mathbf{F}_i \quad \textbf{(4.37)}$$

where $\mathbf{F}_i$ is the net force on particle i.

$$\sum \mathbf{F}_{ext} = M\mathbf{a}_{CM} \qquad \textbf{(4.38)}$$

That is, **the net external force on a system of particles equals the total mass of the system multiplied by the acceleration of the center of mass**. If we compare this with Newton's second law for a single particle, we see that the particle model that we have used for several chapters can be described in terms of the center of mass:

The center of mass of a system of particles of combined mass M moves like an equivalent particle of mass M would move under the influence of the net external force on the system.

Rotation of a Rigid Object About a Fixed Axis

Angular Position, Velocity, and Acceleration

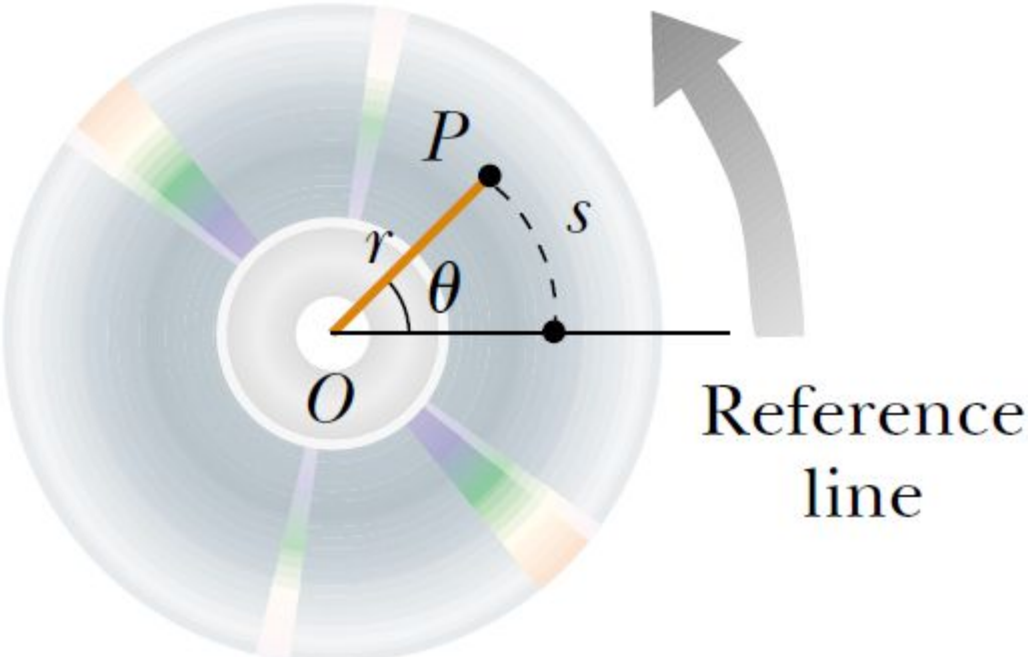


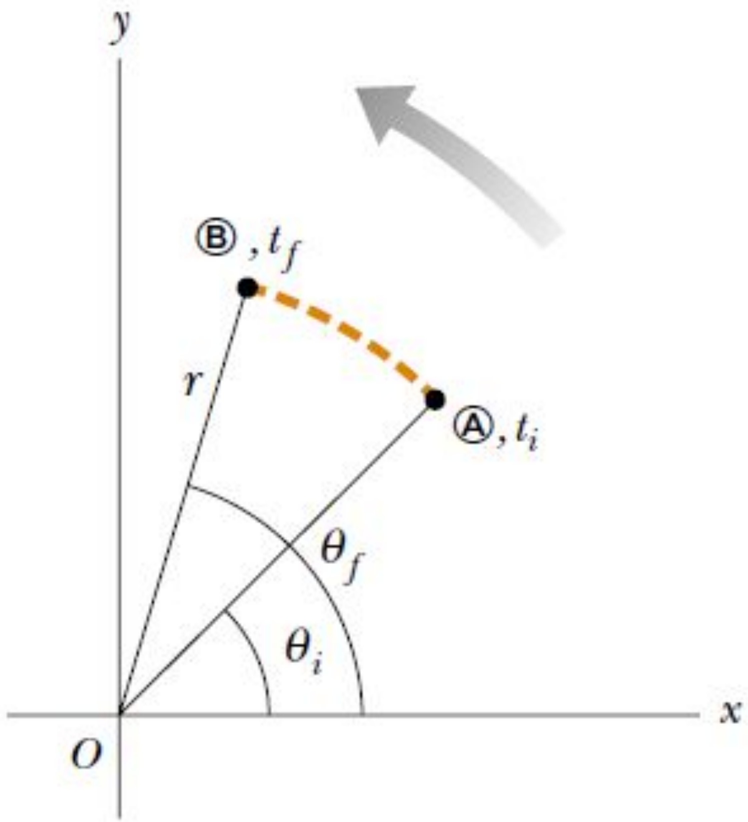
Figure 5.1 A compact disc rotating about a fixed axis through O perpendicular to the plane of the figure.

The angular position of the rigid object is the angle θ between this reference line on the object and the fixed reference line in space, which is often chosen as the *x axis*.

$$\theta = \frac{s}{r} \qquad \textbf{(5.1)}$$

Rotation of a Rigid Object About a Fixed Axis

Angular Position, Velocity, and Acceleration



The average angular speed

$$\bar{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \quad \textbf{(5.2)}$$

The instantaneous angular speed

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \quad \textbf{(5.3)}$$

Figure 5.2 A particle on a rotating rigid object moves from A to B along the arc of a circle. In the time interval $\Delta t = t_f - t_i$, the radius vector moves through an angular displacement $\Delta\theta = \theta_f - \theta_i$.

The average angular acceleration

$$\bar{\alpha} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \qquad \textbf{(5.4)}$$

The instantaneous angular acceleration

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad \textbf{(5.5)}$$

When a rigid object is rotating about a *fixed axis,* **every particle on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration**.

Direction for angular speed and angular acceleration

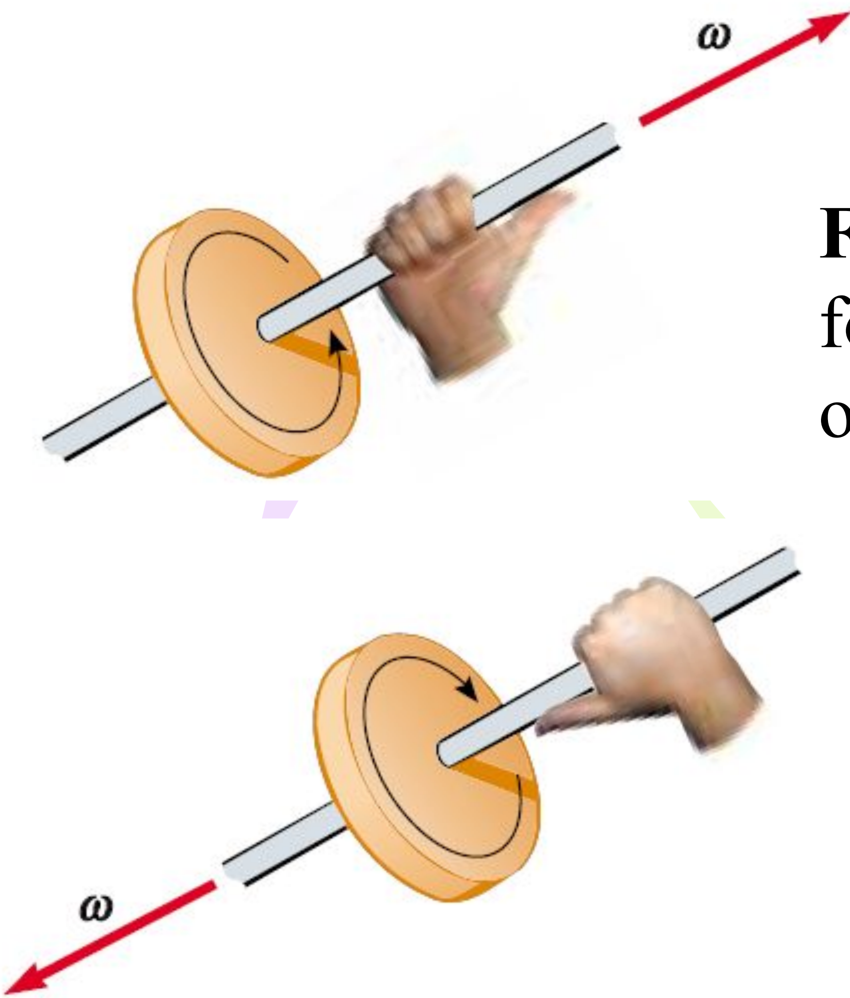


Figure 5.3 The right-hand rule for determining the direction of the angular velocity vector.

Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$

$$d\omega = \alpha \, dt \qquad t_i = 0 \qquad t_f = t$$

$$\omega_f = \omega_i + \alpha t \qquad \text{(for constant } \alpha\text{)} \qquad \textbf{(5.6)}$$

ω_i is the angular speed of the rigid object at time $t = 0$.

Equation 5.6 allows us to find the angular speed ω_f of the object at any later time t.

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \qquad \text{(for constant } \alpha\text{)}$$

(5.7)

θ_i is the angular position of the rigid object at time $t = 0$.

Equation 5.7 allows us to find the angular position θ_f of the object at any later time t.

If we eliminate t from Equations 5.6 and 5.7, we obtain

$$\omega_f^{\,2} = \omega_i^{\,2} + 2\alpha(\theta_f - \theta_i) \qquad \text{(for constant } \alpha\text{)}$$

(5.8)

This equation allows us to find the angular speed ω_f of the rigid object for any value of its angular position θ_f.

If we eliminate α between Equations 5.6 and 5.7, we obtain

$$\theta_f = \theta_i + \tfrac{1}{2}(\omega_i + \omega_f)\,t \qquad \text{(for constant } \alpha\text{)}$$

(5.9)

Table 5.1

Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration	
Rotational Motion About Fixed Axis	**Linear Motion**
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f{}^2 = \omega_i{}^2 + 2\alpha(\theta_f - \theta_i)$	$v_f{}^2 = v_i{}^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Angular and Linear Quantities

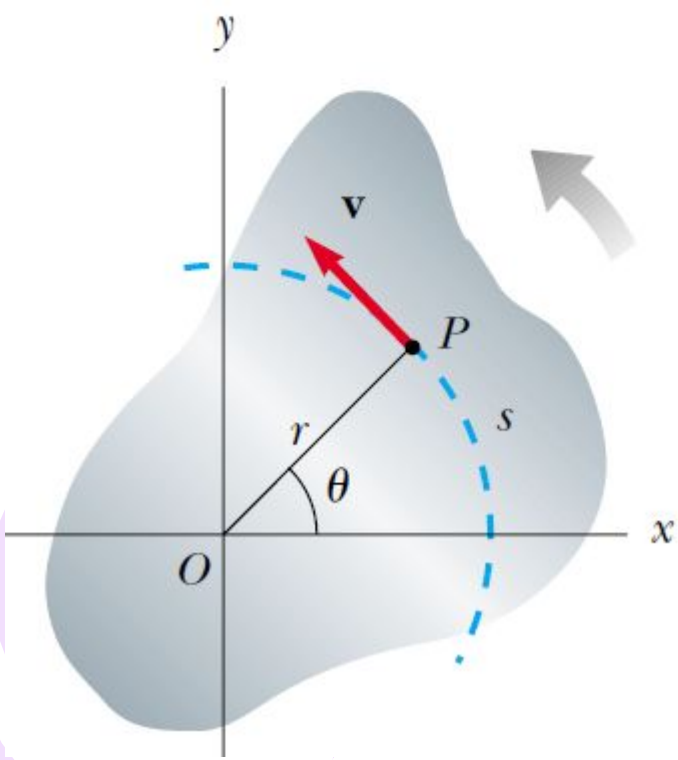


Figure 5.4 As a rigid object rotates about the fixed axis through *O*, the point P has a tangential velocity **v** that is always tangent to the circular path of radius *r*.

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$v = r\omega \qquad \textbf{(5.10)}$$

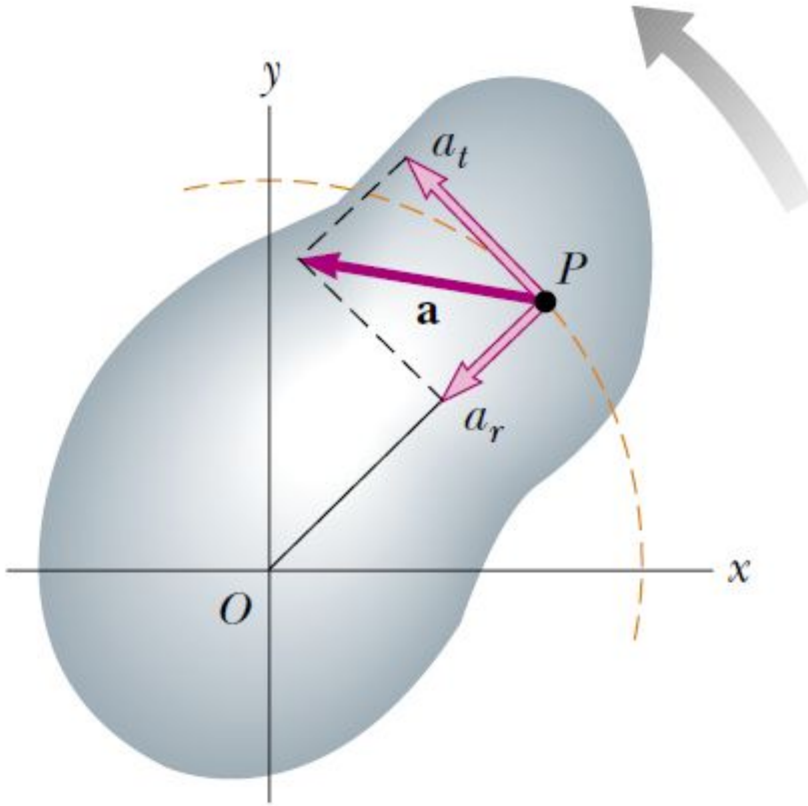
That is, the tangential speed of a point on a rotating rigid object equals the perpendicular distance of that point from the axis of rotation multiplied by the angular speed.

We can relate the angular acceleration of the rotating rigid object to the tangential acceleration of the point P by taking the time derivative of v:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

$$\boxed{a_t = r\alpha} \quad \textbf{(5.11)}$$

That is, the tangential component of the linear acceleration of a point on a rotating rigid object equals the point's distance from the axis of rotation multiplied by the angular acceleration.



A point moving in a circular path undergoes a radial acceleration $\mathbf{a}_r$ of magnitude v^2/r directed toward the center of rotation .

Because v = rω for a point P on a rotating object, we can express the centripetal acceleration at that point in terms of angular speed as

$$a_c = \frac{v^2}{r} = r\omega^2 \quad \textbf{(5.12)}$$

Figure 5.5 As a rigid object rotates about a fixed axis through O, the point P experiences a tangential component of linear acceleration a_t and a radial component of linear acceleration a_r. The total linear acceleration of this point is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$.

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r \quad \textbf{(5.13)}$$

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

Rotational Kinetic Energy

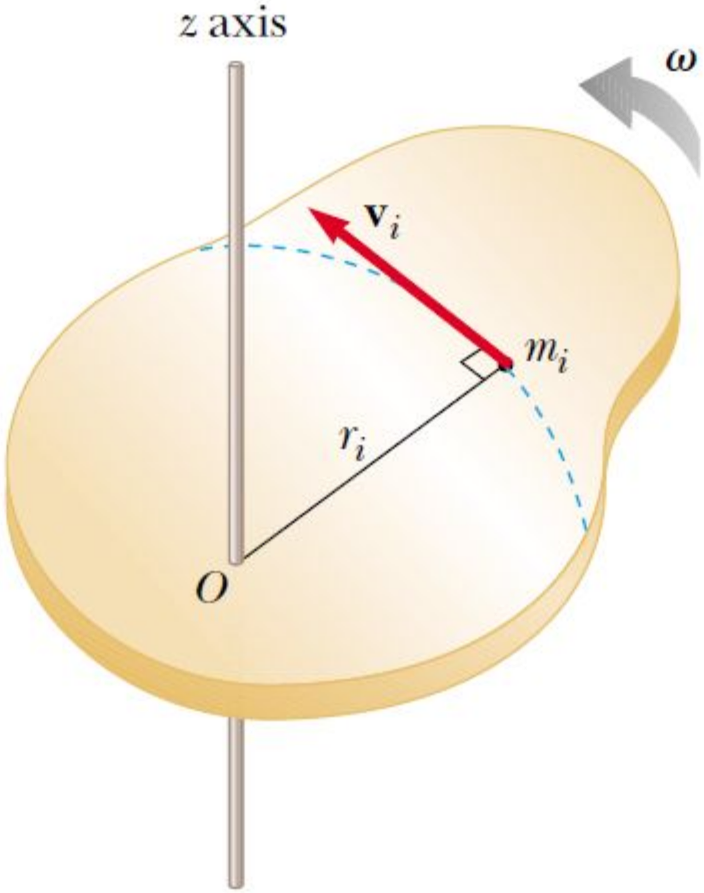


Figure 10.7 A rigid object rotating about the z axis with angular speed ω.

$$K_i = \frac{1}{2} m_i v_i^2$$

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

(5.14)

We simplify this expression by defining the quantity in parentheses as the **moment of inertia I**:

$$I \equiv \sum_i m_i r_i^2 \qquad \textbf{(5.15)}$$

From the definition of moment of inertia, we see that it has dimensions of ML^2 ($kg \cdot m^2$ in SI units). With this notation, Equation 5.14 becomes

$$K_R = \frac{1}{2} I\omega^2 \qquad \textbf{(5.16)}$$

Where K_R is **Rotational kinetic energy**.

Calculation of Moments of Inertia

$$I = \sum_i r_i^{\,2} \Delta m_i \qquad \Delta m_i \to 0$$

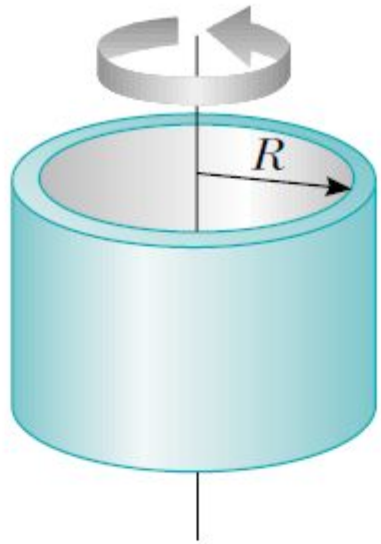
$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^{\,2} \Delta m_i = \int r^2 \, dm \qquad \textbf{(5.17)}$$

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

Table 5.2

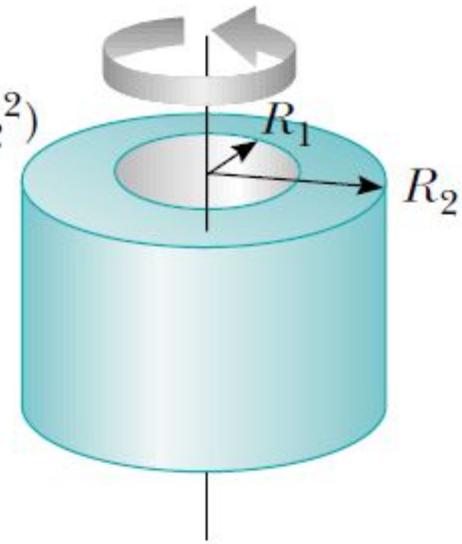
Hoop or thin cylindrical shell

$I_{CM} = MR^2$



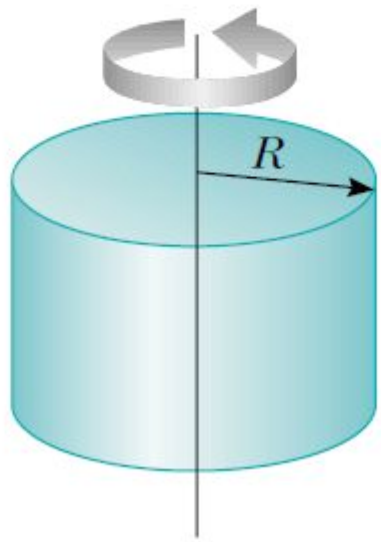
Hollow cylinder

$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$



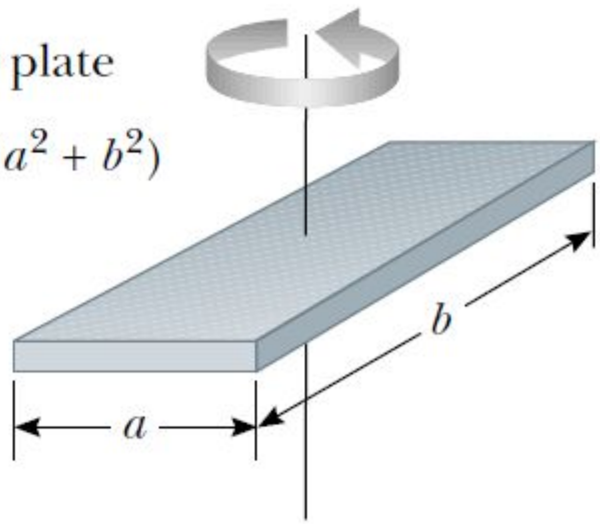
Solid cylinder or disk

$I_{CM} = \frac{1}{2}MR^2$



Rectangular plate

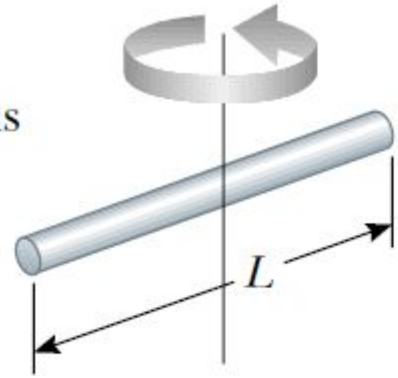
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

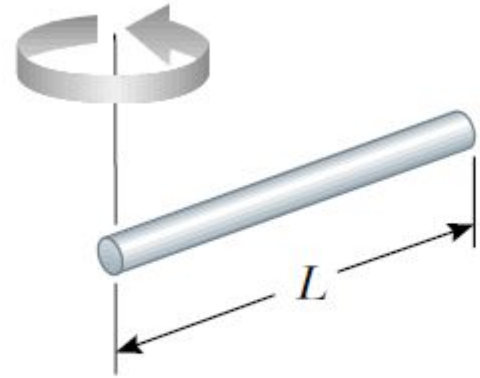
Long thin rod with rotation axis through center

$$I_{CM} = \frac{1}{12} ML^2$$



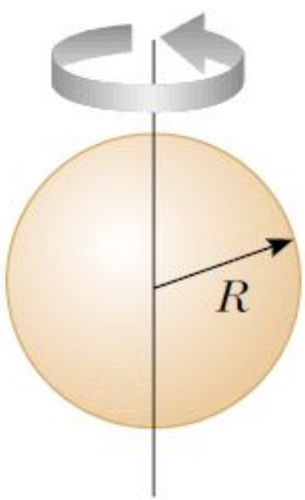
Long thin rod with rotation axis through end

$$I = \frac{1}{3} ML^2$$



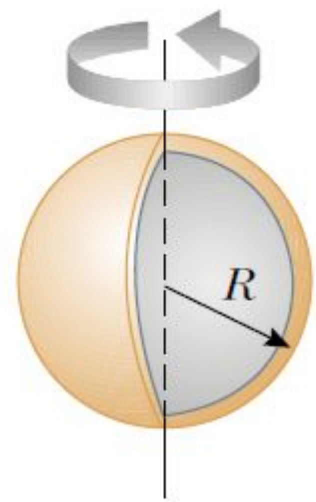
Solid sphere

$$I_{CM} = \frac{2}{5} MR^2$$



Thin spherical shell

$$I_{CM} = \frac{2}{3} MR^2$$



Torque

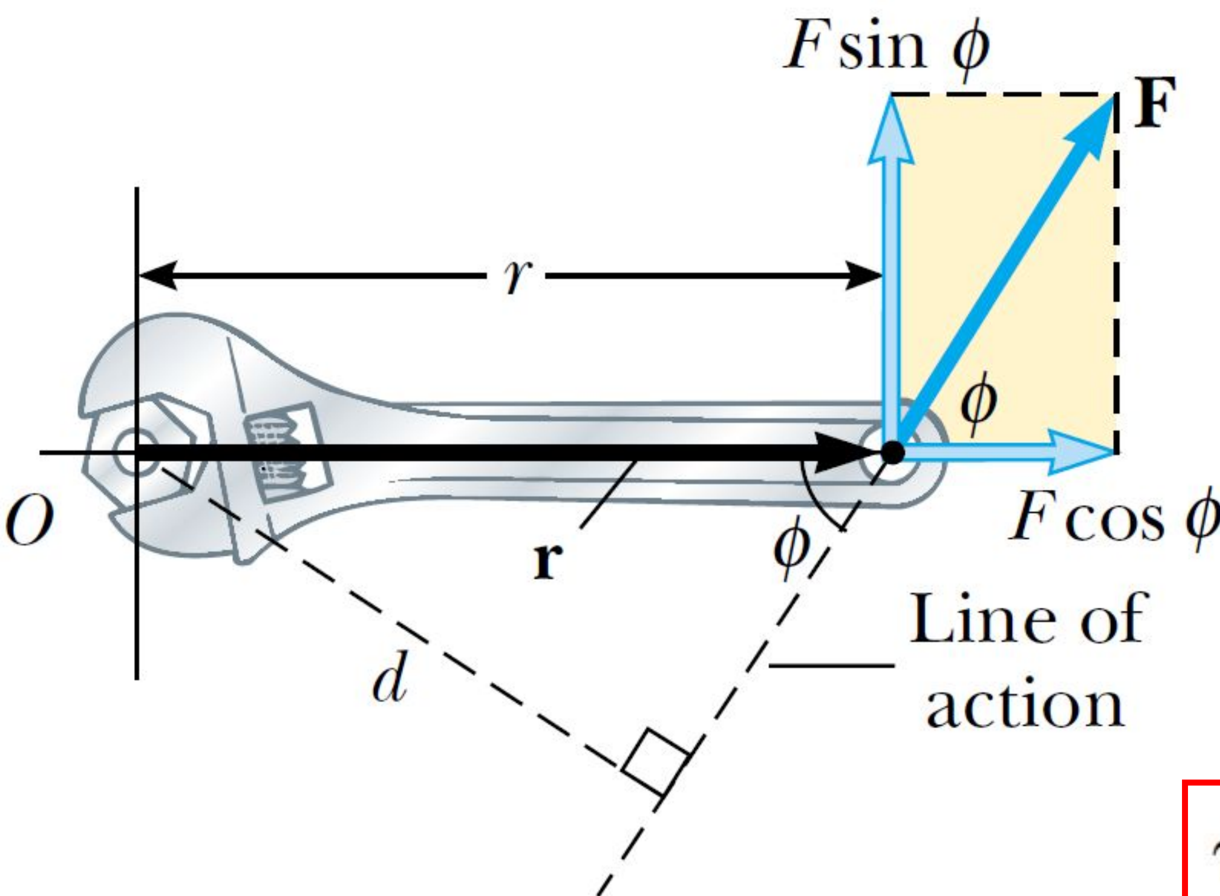


Figure 5.8 The force **F** has a greater rotating tendency about *O* as *F* increases and as the moment arm *d* increases. The component *Fsinφ* tends to rotate the wrench about *O*.

When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by a vector quantity called **torque** τ (Greek tau).

$$\tau \equiv rF\sin\phi = Fd \qquad \textbf{(5.18)}$$

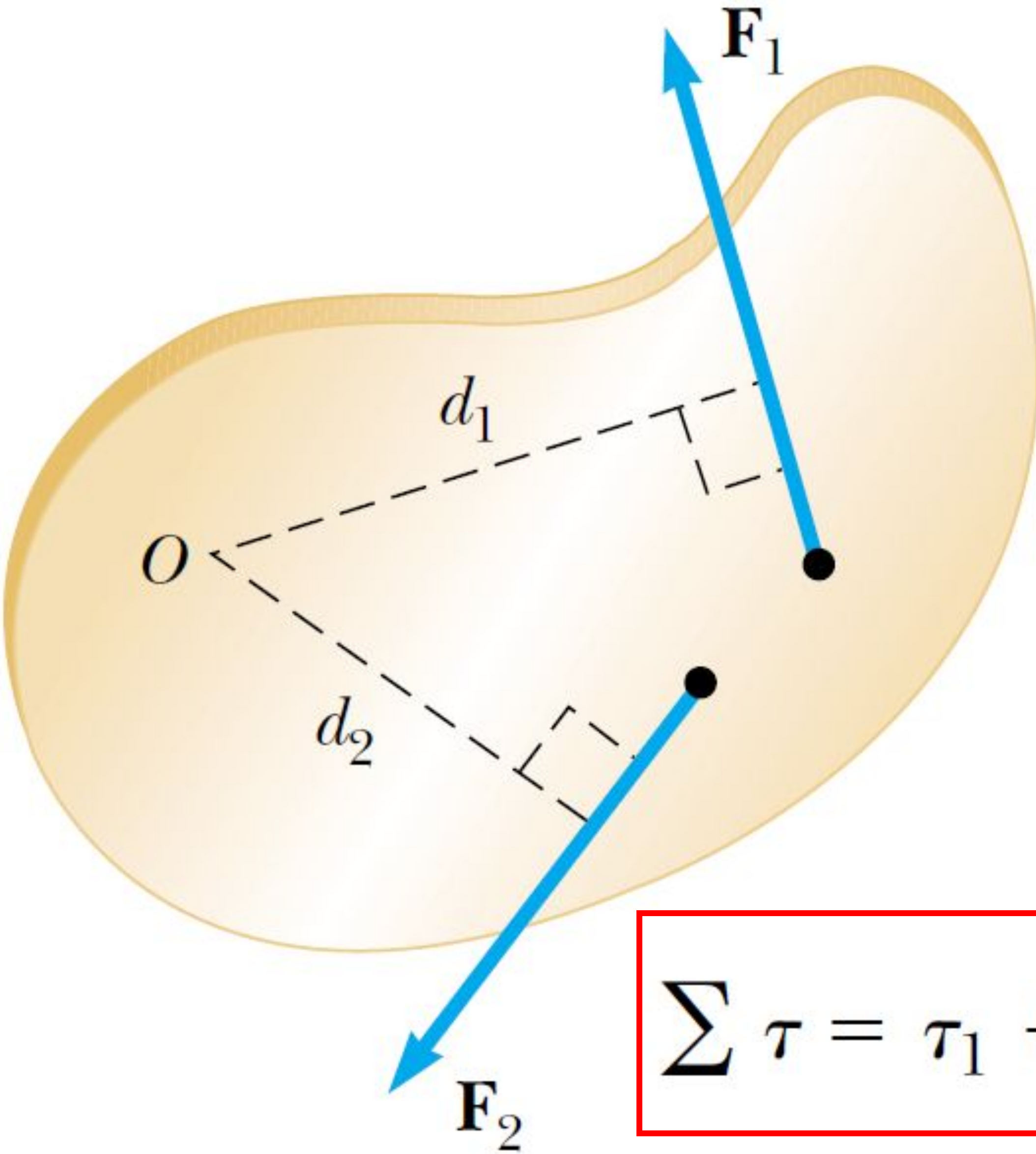
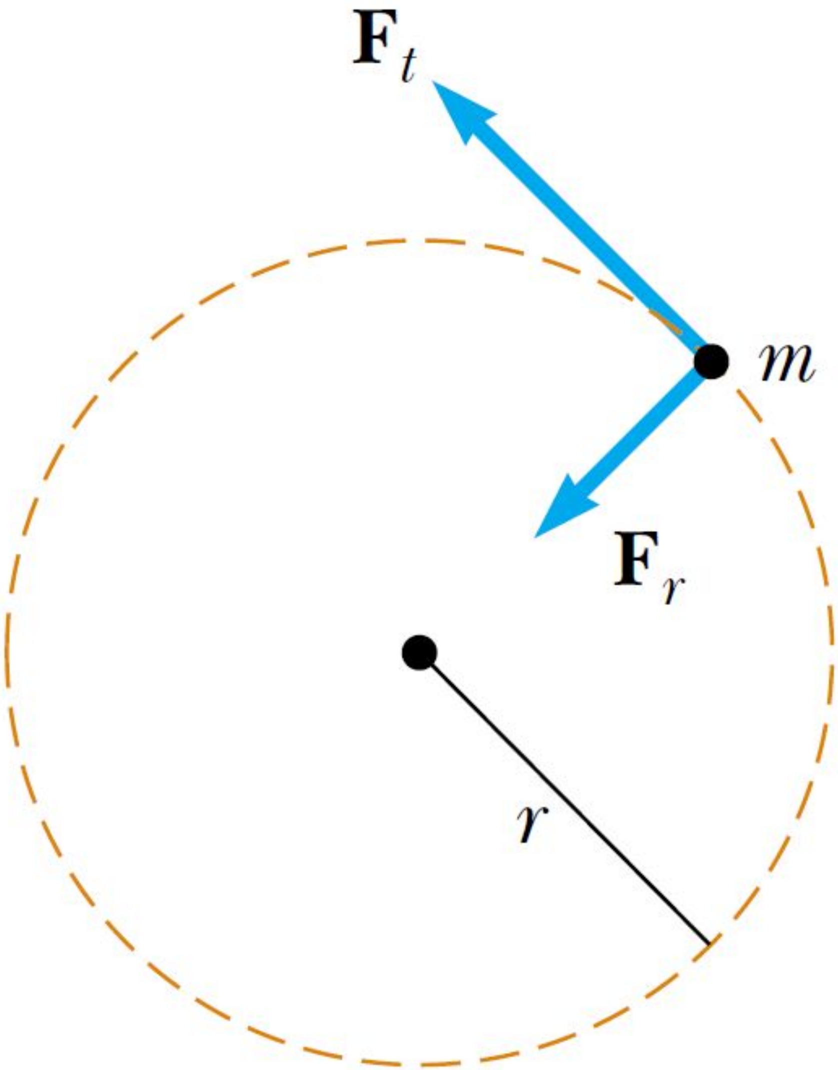


Figure 5.9 The force $\mathbf{F}_1$ tends to rotate the object counterclockwise about *O*, and $\mathbf{F}_2$ tends to rotate it clockwise.

$$\sum \tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$$

(5.19)

Relationship Between Torque and Angular Acceleration



$$F_t = ma_t$$

$$\tau = F_t r = (ma_t) r$$

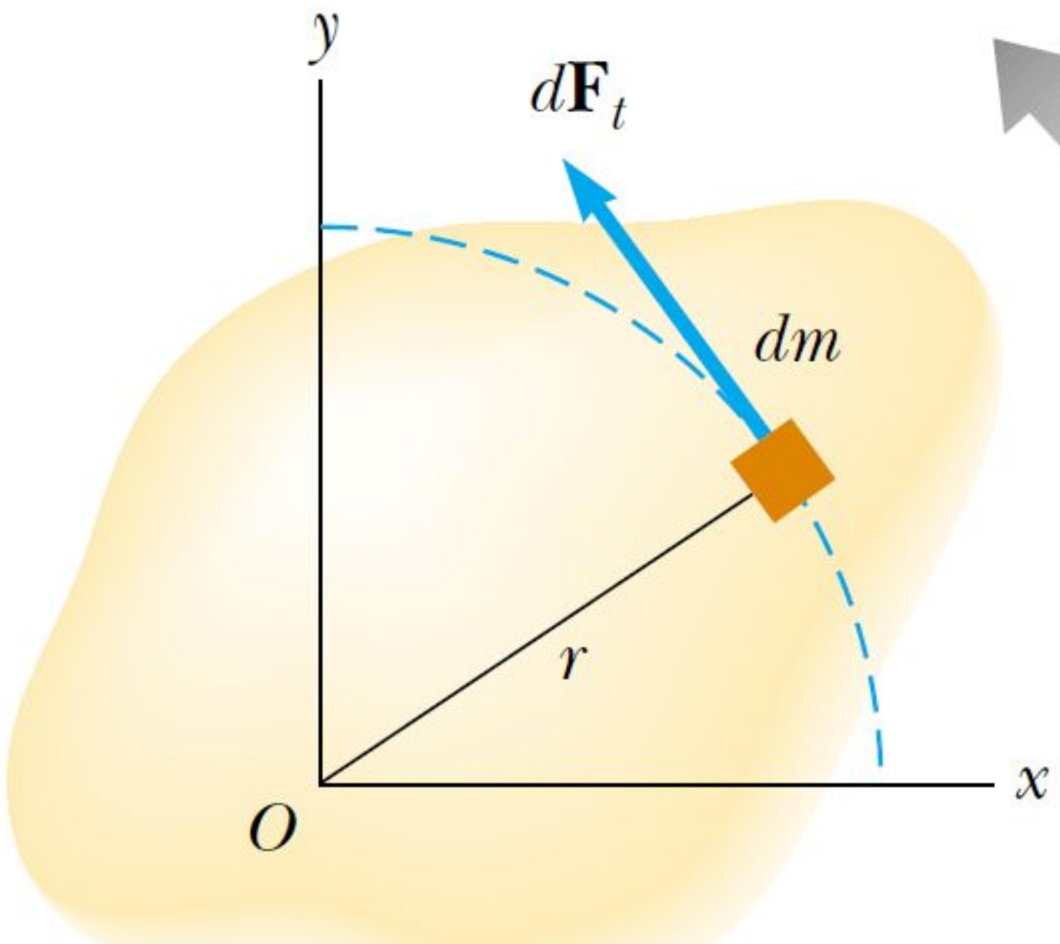
$$a_t = r\alpha$$

Figure 5.10 A particle rotating in a circle under the influence of a tangential force $\mathbf{F}_t$. A force $\mathbf{F}_r$ in the radial direction also must be present to maintain the circular motion.

$$\tau = (mr\alpha)r = (mr^2)\alpha$$

$$\boxed{\tau = I\alpha} \qquad \textbf{(5.20)}$$

The torque acting on the particle is proportional to its angular acceleration, and the proportionality constant is the moment of inertia.



$$dF_t = (dm)\,a_t$$

$$d\tau = r\,dF_t = a_t r\,dm$$

$$a_t = r\alpha$$

$$d\tau = \alpha r^2\,dm$$

Figure 5.11 A rigid object rotating about an axis through *O*.

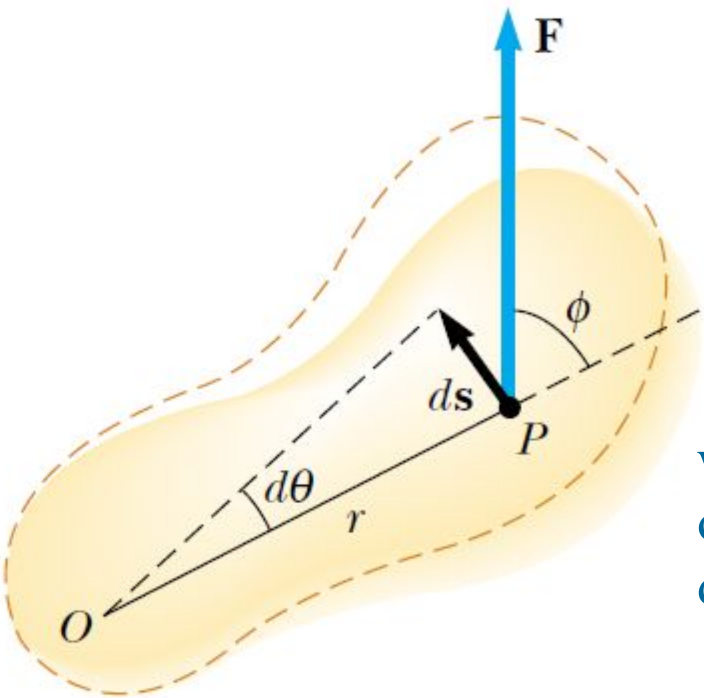
Although each mass element of the rigid object may have a different linear acceleration $\mathbf{a}_t$, they all have the *same* angular acceleration α.

$$\sum \tau = \int \alpha r^2 \, dm = \alpha \int r^2 \, dm$$

$$\sum \tau = I\alpha \qquad \textbf{(5.21)}$$

So, again we see that the net torque about the rotation axis is proportional to the angular acceleration of the object, with the proportionality factor being *I*, a quantity that depends upon the axis of rotation and upon the size and shape of the object.

Work, Power, and Energy in Rotational Motion



The work done by **F** on the object as it rotates through an infinitesimal distance

$$ds = r\,d\theta$$

$$dW = \mathbf{F}\cdot d\mathbf{s} = (F\sin\phi)\,r\,d\theta$$

where *Fsinφ* is the tangential component of **F**, or, in other words, the component of the force along the displacement.

Figure 5.12 A rigid object rotates about an axis through *O* under the action of an external force **F** applied at *P*.

$$dW = \tau\,d\theta \quad \textbf{(5.22)}$$

The rate at which work is being done by **F** as the object rotates about the fixed axis through the angle $d\theta$ in a time interval dt is

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

$$\mathcal{P} = \frac{dW}{dt} = \tau\omega \qquad \textbf{(5.23)}$$

$$\sum \tau = I\alpha$$

$$\sum \tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\omega$$

$$\sum \tau \, d\theta = dW = I\omega \, d\omega$$

$$\sum W = \int_{\omega_i}^{\omega_f} I\omega \, d\omega = \frac{1}{2}I{\omega_f}^2 - \frac{1}{2}I{\omega_i}^2 \qquad \textbf{(5.24)}$$

That is, the work–kinetic energy theorem for rotational motion states that

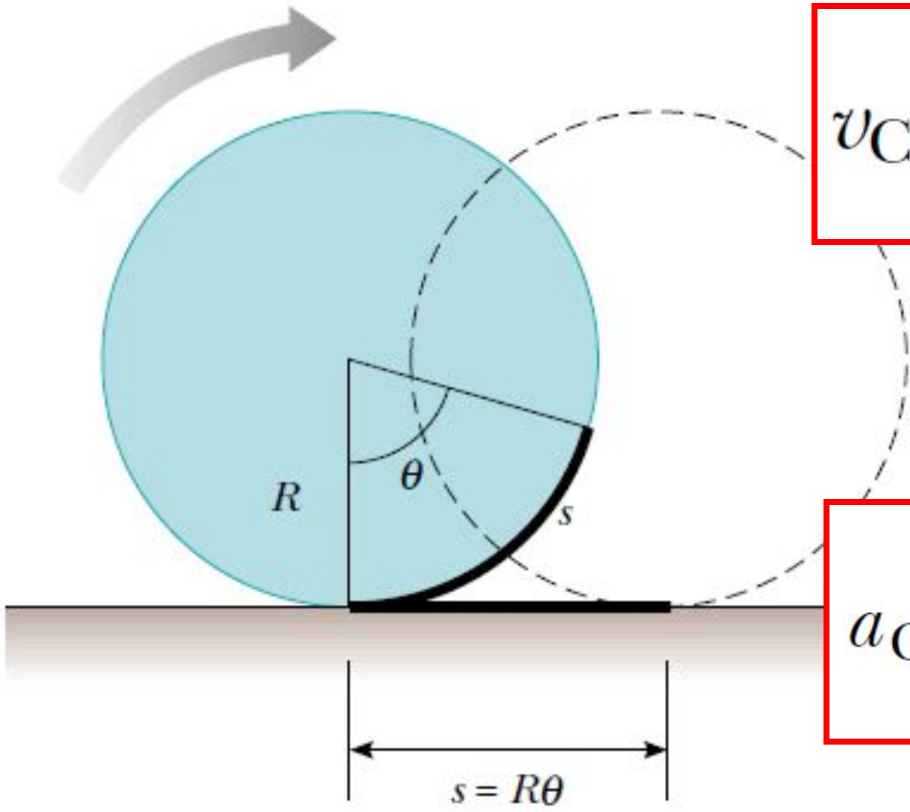
the net work done by external forces in rotating a symmetric rigid object about a fixed axis equals the change in the object's rotational energy.

Table 5.3

Useful Equations in Rotational and Linear Motion

Rotational Motion About a Fixed Axis	Linear Motion
Angular speed $\omega = d\theta/dt$	Linear speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Linear acceleration $a = dv/dt$
Net torque $\Sigma\tau = I\alpha$	Net force $\Sigma F = ma$
If α = constant $\begin{cases} \omega_f = \omega_i + \alpha_t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If a = constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$	Work $W = \int_{x_i}^{x_f} F_x \, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma\tau = dL/dt$	Net force $\Sigma F = dp/dt$

Rolling Motion of a Rigid Object



$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

(5.25)

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$

(5.26)

Figure 5.13 For pure rolling motion, as the cylinder rotates through an angle θ, its center moves a linear distance $s = R\theta$.

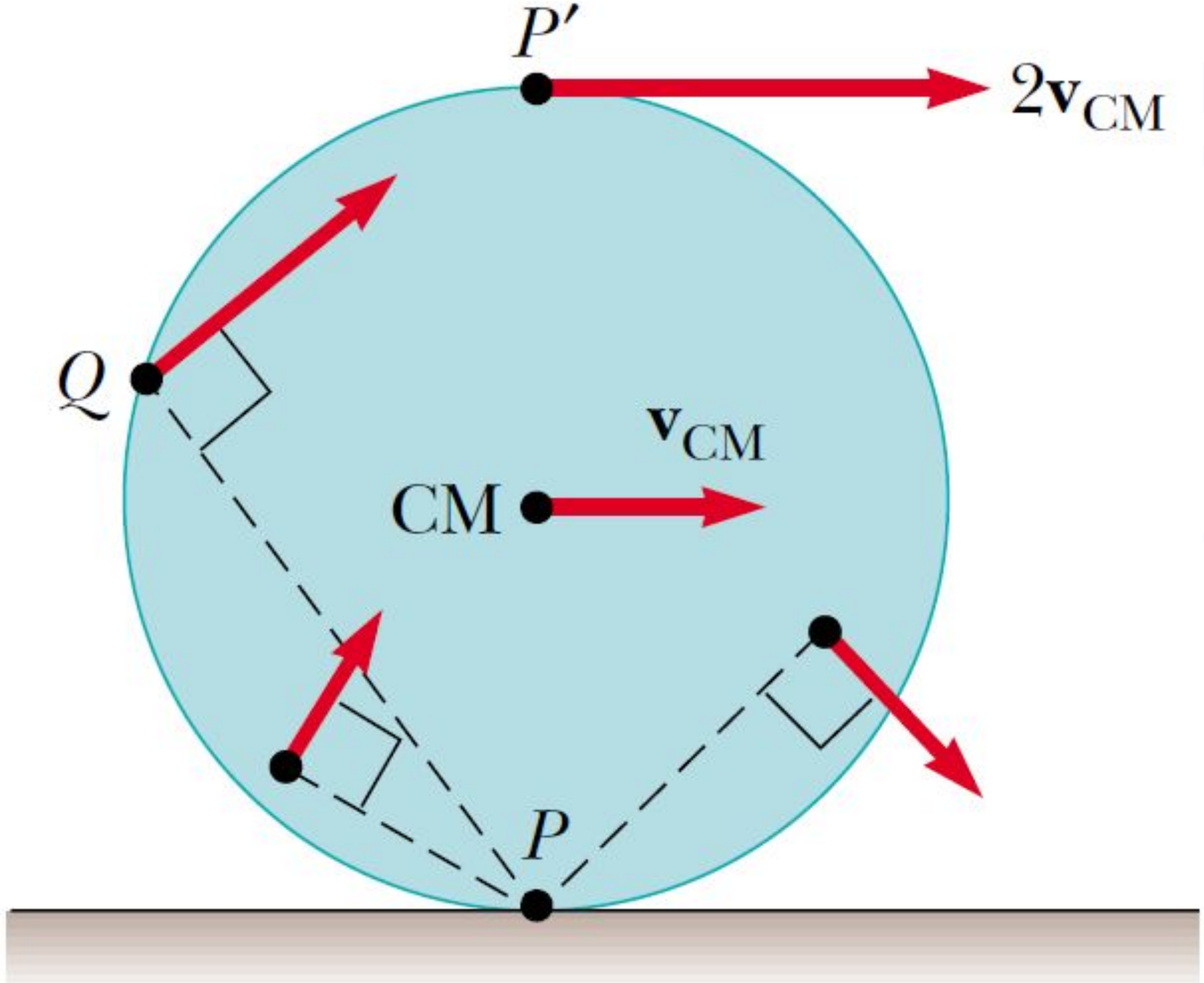


Figure 5.14 All points on a rolling object move in a direction perpendicular to an axis through the instantaneous point of contact P. In other words, all points rotate about P. The center of mass of the object moves with a velocity $\mathbf{v}_{CM}$, and the point P' moves with a velocity 2 $\mathbf{v}_{CM}$.

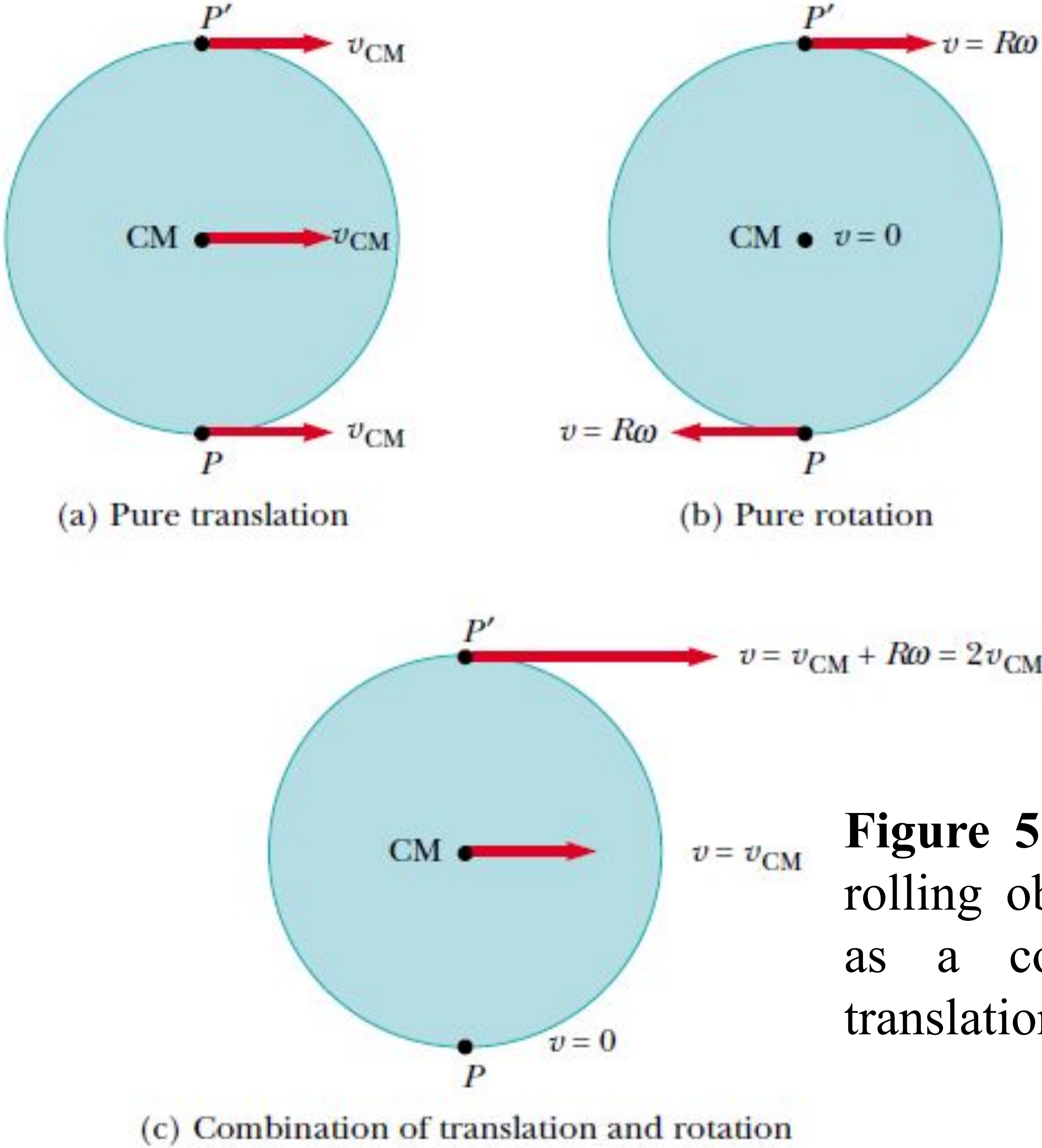


Figure 5.15 The motion of a rolling object can be modeled as a combination of pure translation and pure rotation.

Quiz

Elastic Collisions

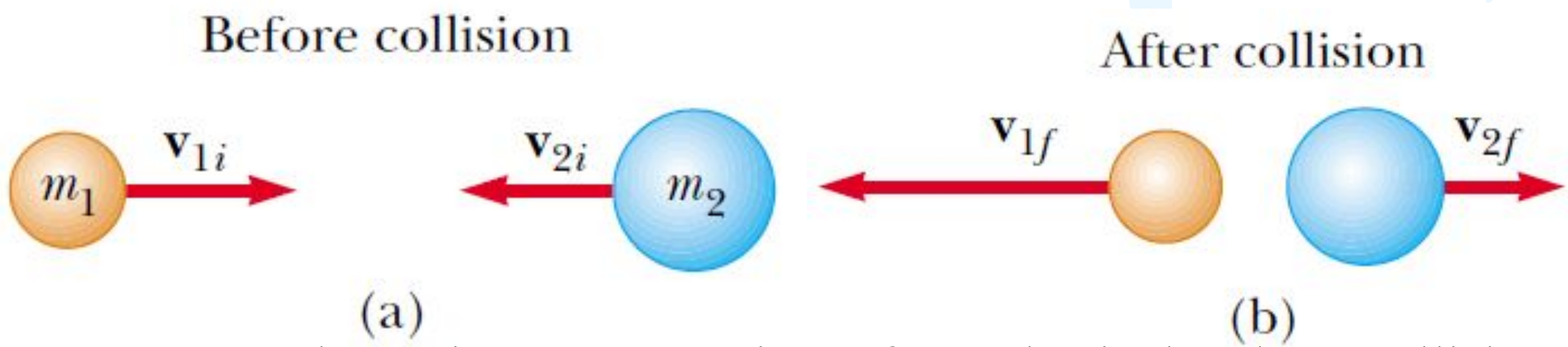


Figure 4.5 Schematic representation of an elastic head-on collision between two particles: (a) before collision and (b) after collision.

Find v_{1f} and v_{2f}.

Quick Quiz 1 A block of mass *m* is projected across a horizontal surface with an initial speed *v*. It slides until it stops due to the friction force between the block and the surface. The same block is now projected across the horizontal surface with an initial speed 2*v*. When the block has come to rest, how does the distance from the projection point compare to that in the first case? (a) It is the same. (b) It is twice as large. (c) It is four times as large. (d) The relationship cannot be determined.

Quick Quiz 2 A car and a large truck traveling at the same speed make a head-on collision and stick together. Which vehicle experiences the larger change in the magnitude of momentum? (a) the car (b) the truck (c) The change in the magnitude of momentum is the same for both. (d) impossible to determine.