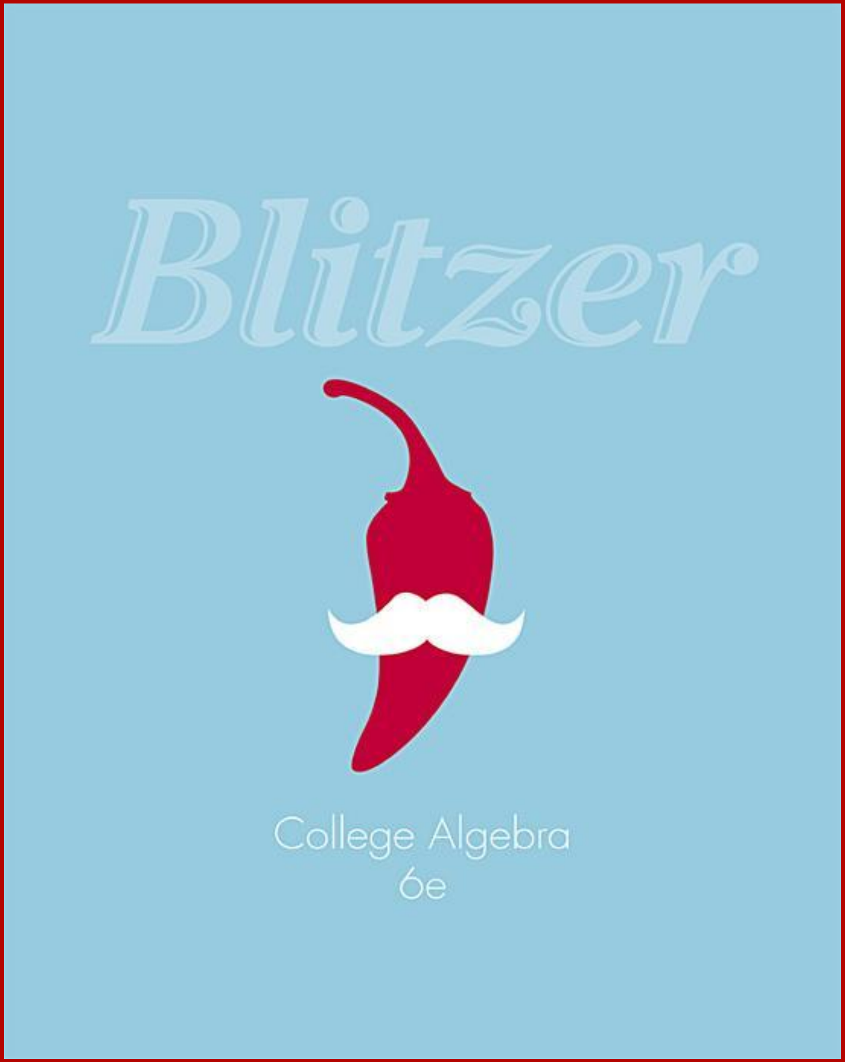
Chapter 3

Polynomial and Rational Functions

3.1 Quadratic Functions



Objectives:

- Recognize characteristics of parabolas.
- Graph parabolas.
- Determine a quadratic function’s minimum or maximum value.
- Solve problems involving a quadratic function’s minimum or maximum value.

The Standard Form of a Quadratic Function

The quadratic function

$$f(x) = a(x-h)^2 + k, \ a \neq 0$$

is in standard form. The graph of f is a parabola whose vertex is the point (h, k). The parabola is symmetric with respect to the line $x = h$. If $a > 0$, the parabola opens upward; if $a < 0$, the parabola opens downward.

Graphing Quadratic Functions with Equations in Standard Form

To graph $f(x) = a(x-h)^2 + k,\ a \neq 0$

1. Determine whether the parabola opens upward or downward. If $a > 0$, it opens upward. If $a < 0$, it opens downward.

2. Determine the vertex of the parabola. The vertex is (h, k).

3. Find any x-intercepts by solving $f(x) = 0$. The function's real zeros are the x-intercepts.

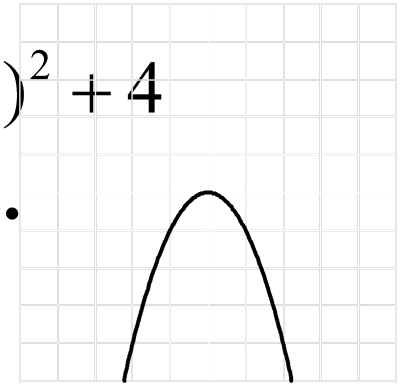
Graphing Quadratic Functions with Equations in Standard Form *(continued)*

To graph $f(x) = a(x-h)^2 + k,\ a \neq 0$

4. Find the y-intercept by computing $f(0)$.

5. Plot the intercepts, the vertex, and additional points as necessary. Connect these points with a smooth curve that is shaped like a bowl or an inverted bowl.

Example: Graphing a Quadratic Function in Standard Form



Graph the quadratic function $f(x) = -(x-1)^2 + 4$

Step 1 Determine how the parabola opens.

$a = -1$, $a < 0$; the parabola opens downward.

Step 2 Find the vertex.

The vertex is at (h, k). Because $h = 1$ and $k = 4$, the parabola has its vertex at (1, 4)

Example: Graphing a Quadratic Function in Standard Form *(continued)*

Graph: $f(x) = -(x-1)^2 + 4$

Step 3 Find the *x*-intercepts by solving *f*(*x*) = 0.

$$f(x) = -(x-1)^2 + 4$$
$$0 = -(x-1)^2 + 4$$
$$-4 = -(x-1)^2$$
$$4 = (x-1)^2$$
$$(x-1) = \pm 2$$

$$x - 1 = 2$$
$$x = 3$$
$$x - 1 = -2$$
$$x = -1$$

The *x*-intercepts are (3, 0) and (–1, 0)

Example: Graphing a Quadratic Function in Standard Form *(continued)*

Graph: $f(x) = -(x-1)^2 + 4$

Step 4 Find the *y*-intercept by computing *f*(0).

$$f(x) = -(x-1)^2 + 4$$

$$f(0) = -(0-1)^2 + 4$$

$$= -(-1)^2 + 4$$

$$= -1 + 4 = 3$$

The *y*-intercept is (0, 3).

Example: Graphing a Quadratic Function in Standard Form

Graph: $f(x) = -(x-1)^2 + 4$

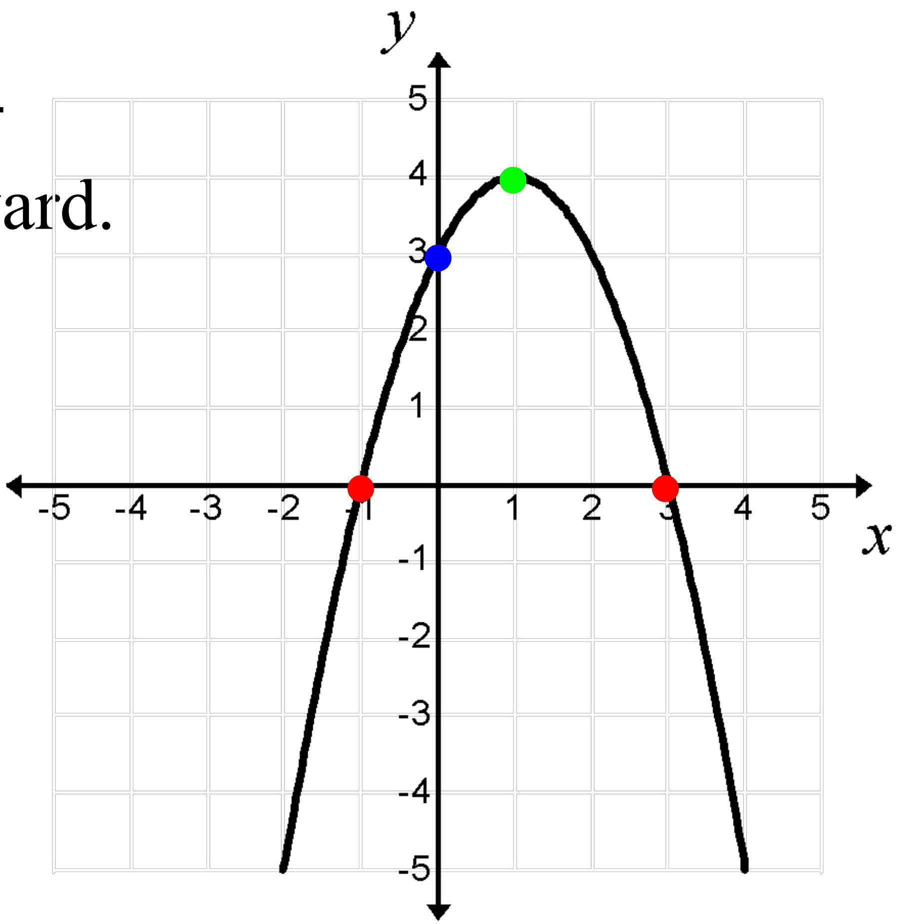
The parabola opens downward.

The *x*-intercepts are

(3, 0) and (–1, 0).

The *y*-intercept is (0, 3).

The vertex is (1, 4).



The Vertex of a Parabola Whose Equation is $f(x) = ax^2 + bx + c$

Consider the parabola defined by the quadratic function

$$f(x) = ax^2 + bx + c.$$

The parabola's vertex is $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right).$

The x-coordinate is $-\dfrac{b}{2a}.$

The y-coordinate is found by substituting the x-coordinate into the parabola's equation and evaluating the function at this value of x.

Graphing Quadratic Functions with Equations in the Form $f(x) = ax^2 + bx + c$

To graph $f(x) = ax^2 + bx + c$,

1. Determine whether the parabola opens upward or downward. If $a > 0$, it opens upward. If $a < 0$, it opens downward.

2. Determine the vertex of the parabola. The vertex is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

3. Find any x-intercepts by solving $f(x) = 0$. The real solutions of $ax^2 + bx + c = 0$ are the x-intercepts.

Graphing Quadratic Functions with Equations in the Form $f(x) = ax^2 + bx + c$ *(continued)*

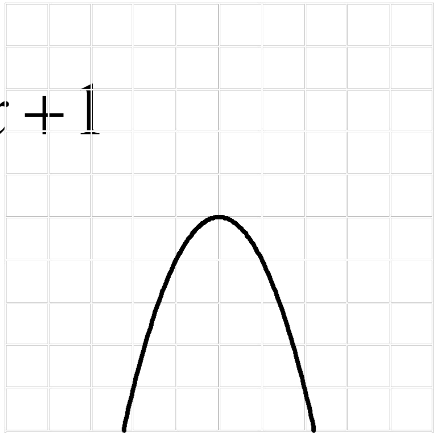
To graph $f(x) = ax^2 + bx + c$

4. Find the *y*-intercept by computing $f(0)$. Because $f(0) = c$ (the constant term in the function's equation), the *y*-intercept is c and the parabola passes through $(0, c)$.

5. Plot the intercepts, the vertex, and additional points as necessary. Connect these points with a smooth curve.

Example: Graphing a Quadratic Function in the Form $f(x) = ax^2 + bx + c$

Graph the quadratic function $f(x) = -x^2 + 4x + 1$



Step 1 Determine how the parabola opens.

$a = -1$, $a < 0$, the parabola opens downward.

Step 2 Find the vertex.

The x-coordinate of the vertex is $x = -\frac{b}{2a}$.

$a = -1$, $b = 4$, and $c = 1$

$$x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$$

Example: Graphing a Quadratic Function in the Form $f(x) = ax^2 + bx + c$ *(continued)*

Graph: $f(x) = -x^2 + 4x + 1$

Step 2 ***(continued)*** **find the vertex.**

The coordinates of the vertex are $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

We found that $x = 2$ at the vertex.

$$f(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$$

The coordinates of the vertex are (2, 5).

Example: Graphing a Quadratic Function in the Form $f(x) = ax^2 + bx + c$ (continued)

Graph: $f(x) = -x^2 + 4x + 1$

Step 3 Find the *x*-intercepts by solving *f*(*x*) = 0.

$$f(x) = -x^2 + 4x + 1 \rightarrow 0 = -x^2 + 4x + 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{(4)^2 - 4(-1)(1)}}{2(-1)} = \frac{-4 \pm \sqrt{16+4}}{-2}$$

$$= \frac{-4 \pm \sqrt{20}}{-2} \quad x = \frac{-4 + \sqrt{20}}{-2} \approx -0.2 \quad x = \frac{-4 - \sqrt{20}}{-2} \approx 4.2$$

The *x*-intercepts are (–0.2, 0) and (4.2, 0).

Example: Graphing a Quadratic Function in the Form $f(x) = ax^2 + bx + c$ *(continued)*

Graph: $f(x) = -x^2 + 4x + 1$

Step 4 Find the *y*-intercept by computing f(0).

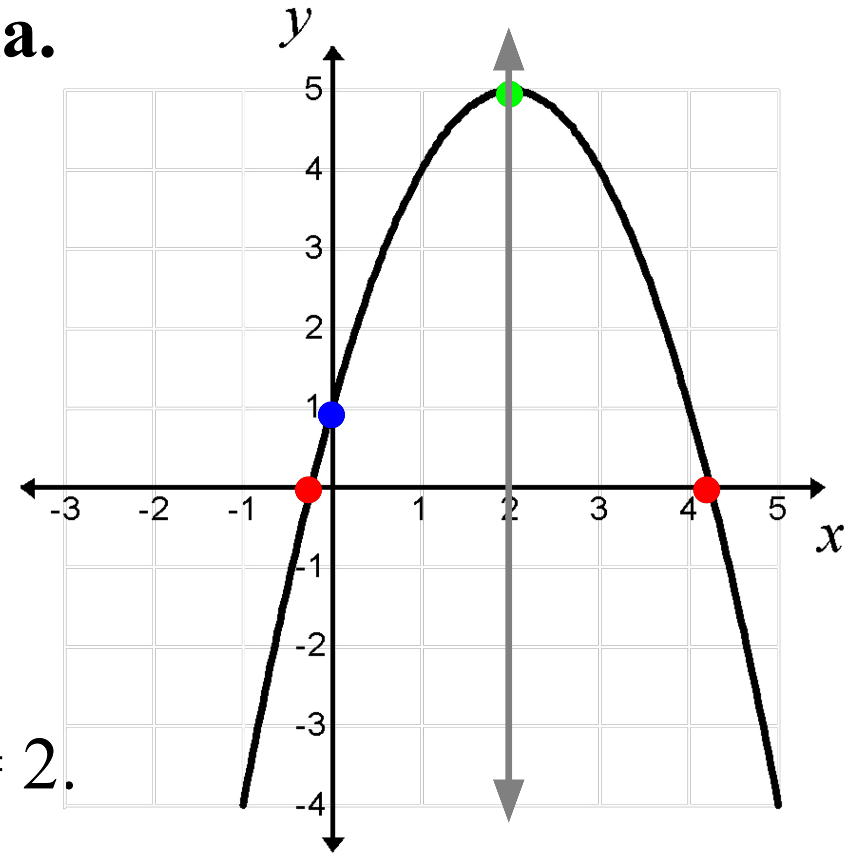
$$f(0) = -(0)^2 + 4(0) + 1 = 1$$

The *y*-intercept is (0, 1).

Example: Graphing a Quadratic Function in the Form $f(x) = ax^2 + bx + c$ (continued)

Graph: $f(x) = -x^2 + 4x + 1$

Step 5 Graph the parabola.



The x-intercepts are

(–0.2, 0) and (4.2, 0).

The y-intercept is (0, 1).

The vertex is (2, 5).

The axis of symmetry is $x = 2$.

Minimum and Maximum: Quadratic Functions

Consider the quadratic function $f(x) = ax^2 + bx + c.$

1. If $a > 0$, then f has a minimum that occurs at $x = -\frac{b}{2a}$

This minimum value is $f\left(-\frac{b}{2a}\right).$

2. If $a < 0$, then f has a maximum that occurs at $x = -\frac{b}{2a}$

This maximum value is $f\left(-\frac{b}{2a}\right).$

Minimum and Maximum: Quadratic Functions *(continued)*

Consider the quadratic function $f(x) = ax^2 + bx + c.$

In each case, the value of $x = -\dfrac{b}{2a}$ gives the location of the minimum or maximum value.

The value of y, or $f\left(-\dfrac{b}{2a}\right)$ gives that minimum or maximum value.

Example: Obtaining Information about a Quadratic Function from Its Equation

Consider the quadratic function $f(x) = 4x^2 - 16x + 1000$

Determine, without graphing, whether the function has a minimum value or a maximum value.

$a = 4;\ \ a > 0.$

The function has a minimum value.

Example: Obtaining Information about a Quadratic Function from Its Equation *(continued)*

Consider the quadratic function $f(x) = 4x^2 - 16x + 1000$

Find the minimum or maximum value and determine where it occurs.

$a = 4,\ b = -16,\ c = 1000$

$$x = -\frac{b}{2a} = -\frac{-16}{2(4)} = \frac{16}{8} = 2$$

$$f\left(-\frac{b}{2a}\right) = f(2) = 4(2)^2 - 16(2) + 1000 = 16 - 32 + 1000$$
$$= 984$$

The minimum value of *f* is 984.

This value occurs at $x = 2$.

Example: Obtaining Information about a Quadratic Function from Its Equation

Consider the quadratic function $f(x) = 4x^2 - 16x + 1000$

Identify the function's domain and range (without graphing).

Like all quadratic functions, the domain is $(-\infty, \infty)$.

We found that the vertex is at (2, 984).

$a > 0$, the function has a minimum value at the vertex.

The range of the function is $(984, \infty)$.

Strategy for Solving Problems Involving Maximizing or Minimizing Quadratic Functions

1. Read the problem carefully and decide which quantity is to be maximized or minimized.
2. Use the conditions of the problem to express the quantity as a function in one variable.
3. Rewrite the function in the form $f(x) = ax^2 + bx + c$.

Strategy for Solving Problems Involving Maximizing or Minimizing Quadratic Functions *(continued)*

4. Calculate $-\frac{b}{2a}$. If $a > 0$, f has a minimum at $x = -\frac{b}{2a}$. This minimum value is $f\left(-\frac{b}{2a}\right)$. If $a < 0$, f has a maximum at $x = -\frac{b}{2a}$. This maximum value is $f\left(-\frac{b}{2a}\right)$.

5. Answer the question posed in the problem.

Example: Maximizing Area

You have 120 feet of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. What is the maximum area?

Step 1 Decide what must be maximized or minimized

We must maximize area.

Example: Maximizing Area *(continued)*

Step 2 Express this quantity as a function in one variable.

We must maximize the area of the rectangle, $A = xy$.

We have 120 feet of fencing, the perimeter of the rectangle is 120. $2x + 2y = 120$

Solve this equation for y:

$$2y = 120 - 2x$$

$$y = \frac{120 - 2x}{2} = 60 - x$$

$$A = xy$$

$$A = x(60 - x)$$

$$A = 60x - x^2$$

$$A = -x^2 + 60x$$

Example: Maximizing Area *(continued)*

Step 3 Write the function in the form

$$f(x) = ax^2 + bx + c$$

$$A(x) = -x^2 + 60x$$

Step 4 Calculate $-\frac{b}{2a}$ $\quad x = -\frac{b}{2a} = -\frac{60}{2(-1)} = 30$

$a < 0$, so the function has a maximum at this value.

This means that the area, $A(x)$, of a rectangle with perimeter 120 feet is a maximum when one of the rectangle's dimensions, x, is 30 feet.

Example: Maximizing Area *(continued)*

Step 5 Answer the question posed by the problem.

You have 120 feet of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. What is the maximum area?

The rectangle that gives the maximum square area has dimensions 30 ft by 30 ft.

$$A(x) = -x^2 + 60x \rightarrow A(30) = -(30)^2 + 60x \ = 900$$

The maximum area is 900 square feet.