NUFYP Mathematics

4.2 Introduction to Limits

Inês Russinho Mouga

Lecture Outline

- **Sequences**
  - Limits
  - Convergent
  - Divergent
- **Calculating Limits**
  - Numerically
  - Algebraically
  - Graphically (next lecture)

Introduction

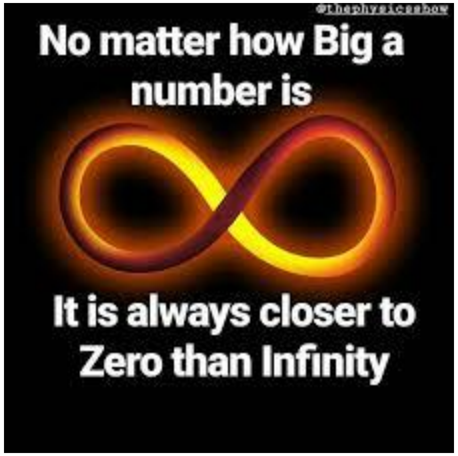
The concept of a limit is the fundamental building block on which most calculus concepts are based.

As you saw in the preview activity's video, a group of atoms is so small that its size is almost

Except that it's not zero.
Galaxy Group is 10 million
y from us. This distance so
we find it hard to imagine. But
asure it.

Introduction

There is no end to the natural numbers. Say the largest number you can imagine, the colleague sitting next to you will say the same number plus one, and that will be a larger number. And we can go on and on forever…



Infinitely large…

- Let's consider the sequence of even numbers

$$u_n = 2n$$

Some of its terms are

$$2, 4, 6, 8, 10, \ldots$$

Is there a number larger than all other? Of course not. No matter how large is the number we pick, the following even numbers will all be larger than that, and larger than all other even numbers before that one.

What is the domain of u_n?

- We say that the sequence of even numbers $u_n = 2n$ goes to positive infinity as n becomes large, and we write

$$\lim_{n \to +\infty} u_n = +\infty$$

This expression can be read as “the limit of u_n as n approaches positive infinity is positive infinity”.

Intuitively, it means that as n becomes very large and positive, u_n also becomes very large and positive.

- Similarly, we can conclude that the sequence of negative even numbers

$$v_n = -2n$$

goes to negative infinity as n becomes large. We write

$$\lim_{n \to +\infty} v_n = -\infty$$

How would you read this expression?

Note: when it is **clear** that we are referring to a limit at positive infinity, we can omit the **+** sign.

- Let's now consider the sequence

$$w_n = (-1)^n 2n$$

Some of its terms are

$$-2, 4, -6, 8, -10, \ldots$$

As n becomes large, will it go towards positive infinity or negative infinity? We cannot say, because the signs keep alternating, - + - + - + - + - + - + - + - +…

However, we can say that in **module**, i.e. taking the absolute value of each term, it goes to positive infinity.

- Because n is a natural number, all the following sequences tend to infinity (i.e. each term is larger than the previous one):

$$a_n = n$$
$$b_n = \sqrt{n}$$
$$c_n = n^2$$
$$d_n = 2^n$$

Can we generalise the sequence d_n to include other values?

- The sequence $d_n = 2^n$ is similar to the exponential function $f(x) = 2^x$, which as you recall, increases without bound.

- Any sequence of the type $b_n = a^n, a > 1$, will tend to infinity.
- If $a < 1$, then $b_n = a^n$ will tend to 0.

- If $u_n \to +\infty$, then $u_n + \alpha \to +\infty$, $\forall \alpha \in \mathbb{R}$
- If $u_n \to +\infty$ and $\lambda > 0$, then $\lambda u_n \to +\infty$

Why do we need $\lambda > 0$?

Example 1: Compute the limits of the following sequences, using the previous sequences / properties to justify your answer.

1. $a_n = \sqrt{n} - 5$
2. $b_n = n - \sqrt{n}$
3. $c_n = \dfrac{3^{2n}}{5^n}$
4. $d_n = \left(-\dfrac{3}{2}\right)^n$

Solution:

1. $a_n = \sqrt{n} - 5$

We know that $\lim_{n \to +\infty} \sqrt{n} = +\infty$. If we subtract a real number from an infinitely large number, we still obtain an infinitely large number. Therefore,

$$\lim_{n \to +\infty} (\sqrt{n} - 5) = +\infty$$

2. $b_n = n - \sqrt{n}$

We know that $\lim_{n\to+\infty} n = +\infty$ and $\lim_{n\to+\infty} \sqrt{n} = +\infty$.

If we subtract ∞ from ∞, we cannot be sure what we will obtain. Are they equally large? Is one greater than the other?

Is $+\infty - \infty$ equal to: $+\infty$?

$-\infty$?

0?

We do not know. A situation like this is called an **intedermination**.

- We need to rewrite the expression in a way that allows us overcome this obstacle:

$$b_n = n - \sqrt{n} = \sqrt{n}(\sqrt{n} - 1)$$

Therefore, we have

$$\lim_{n \to +\infty} \sqrt{n}(\sqrt{n} - 1) = +\infty(+\infty) = +\infty$$

Try analysing the limits by considering which term “grows faster”.

3. $c_n = \dfrac{3^{2n}}{5^n}$

$$\frac{3^{2n}}{5^n} = \left(\frac{3^2}{5}\right)^n = \left(\frac{9}{5}\right)^n$$

This is a sequence of the type we saw before, so, because $\frac{9}{5} > 1$,

$$\lim_{n\to+\infty} c_n = +\infty$$

4. $d_n = \left(-\frac{3}{2}\right)^n$

$$\left(-\frac{3}{2}\right)^n = (-1)^n\left(\frac{3}{2}\right)^n$$

We know that $\lim_{n\to+\infty}\left(\frac{3}{2}\right)^n = +\infty$, but as the $(-1)^n$ alternates the sign of the terms, we cannot conclude whether the sequence tends to $+\infty$ or $-\infty$.

We can say that in **module** $\lim_{n\to+\infty}|d_n| = +\infty$

Your turn!

- Compute the limits of the following sequences, using the previous properties to justify your answer.

1. $a_n = 2^n + 5\sqrt{n}$
2. $b_n = 3n^2 - 6n$
3. $c_n = \frac{2^n - 1}{3}$
4. $d_n = \frac{n^2 - 2n}{2n - 1}$

Your turn!

1. $2^n \to +\infty, \sqrt{n} \to +\infty, 5\sqrt{n} \to +\infty,$ $\boldsymbol{a_n} \to +\infty$

2. $n \to +\infty$, $3n - 6 \to +\infty$, $\boldsymbol{b_n} \to +\infty$

3. $c_n = \frac{1}{3}(2^n - 1),\ 2^n - 1 \to +\infty,$ $\boldsymbol{c_n} \to +\infty$

4. $d_n = \frac{n^2-2n}{2n-1} > \frac{n^2-2n}{2n} = \frac{n(n-2)}{2n} = \frac{n-2}{2},$ if $n > 1.$

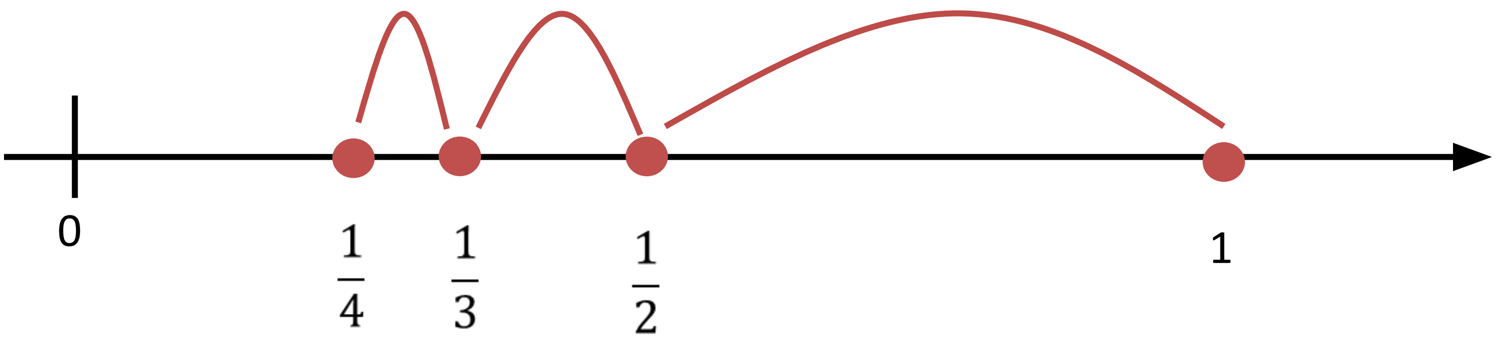
$\boldsymbol{d_n} \to +\infty$

Infinitely small…

- Let's consider the sequence

$$u_n = \frac{1}{n}$$

Some of the terms of u_n are represented on the real line below

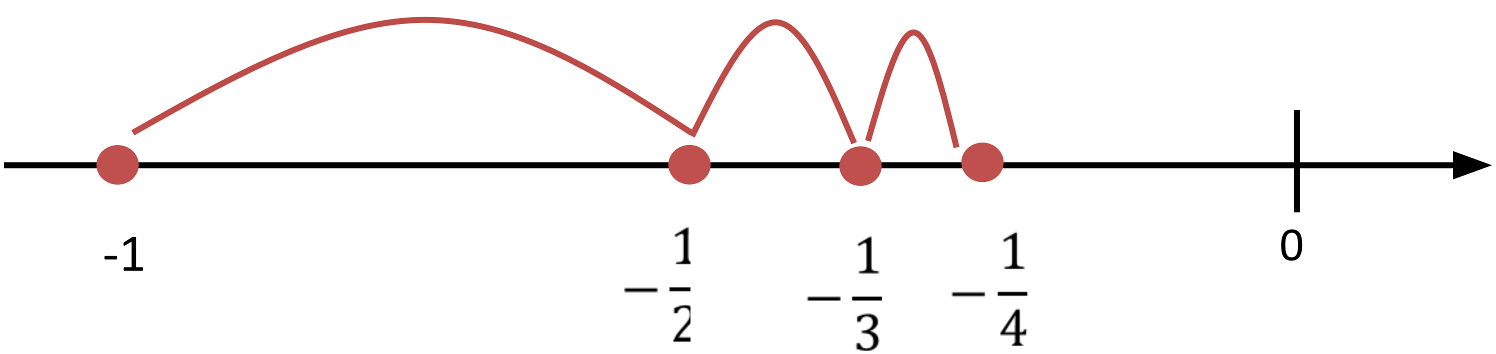


Infinitely small…

- Consider also the sequence

$$w_n = -\frac{1}{n}$$

Some of its terms are represented on the real line below

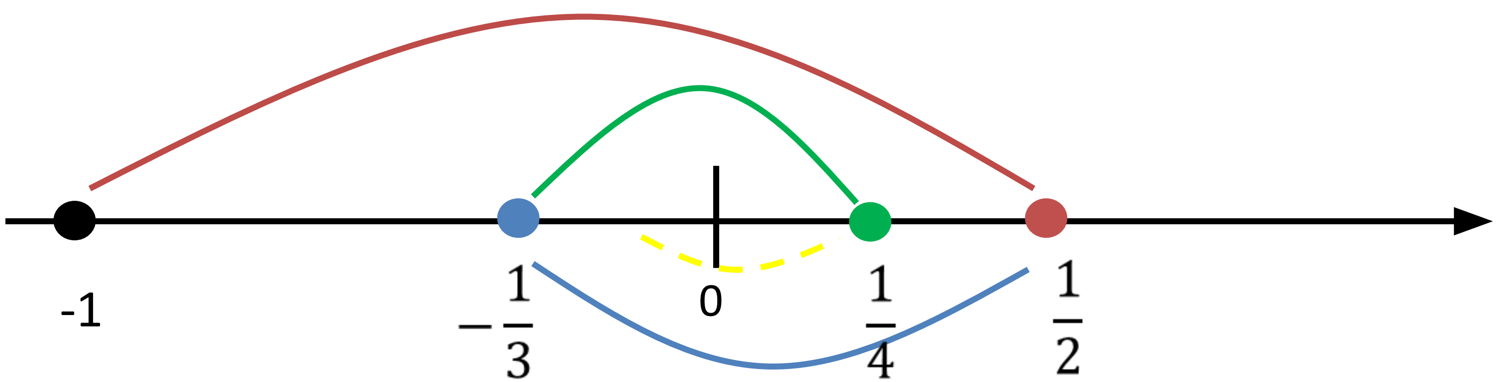


Infinitely small…

And finally, consider the sequence

$$z_n = (-1)^n \frac{1}{n}$$

Some of its terms are represented on the real line below



- In all three cases, the sequences **tend to zero**, even though they will **never reach that value**. We write

$$\lim_{n\to+\infty} u_n = 0, \qquad \lim_{n\to+\infty} w_n = 0, \qquad \lim_{n\to+\infty} z_n = 0$$

or

$$\frac{1}{n} \to 0\,, \qquad -\frac{1}{n} \to 0, \qquad (-1)^n\frac{1}{n} \to 0$$

What's the difference between sequences $z_n = (-1)^n\frac{1}{n}$ and $w_n = (-1)^n 2n$? Why can we compute one limit but not the other?

Rules

- All the following sequences tend to zero:

$$a_n = \frac{1}{n} \qquad b_n = \frac{1}{\sqrt{n}}$$

$$c_n = \frac{1}{n^2} \qquad d_n = \frac{1}{2^n}$$

- If $u_n \to 0$ and $\lambda \in \mathbb{R}$, then $\lambda u_n \to 0$
- If $u_n \to +\infty$, then $\frac{1}{u_n} \to 0$, if $u_n \neq 0 \ \forall n \in \mathbb{N}$
- If $u_n \to 0$, then $\frac{1}{u_n} \to \infty$, if $u_n \neq 0 \ \forall n \in \mathbb{N}$
- If $-1 < a < 1$, then $a^n \to 0$

Example 2: Show that the following sequences tend to zero, using the previous properties to justify your answer.

1. $a_n = \frac{4}{2^n}$
2. $b_n = \frac{n}{n^2+2}$
3. $c_n = \frac{5^n}{3^{2n}}$

Solution:

1. $a_n = \frac{4}{2^n}$

We know that $\lim\limits_{n\to+\infty} \frac{1}{2^n} = 0$. Therefore,

$$\lim_{n\to+\infty} 4 \cdot \frac{1}{2^n} = 0$$

2. $b_n = \frac{n}{n^2+2}$

If we just substitute n, we end up with an **indetermination**:

$$\lim_{n\to+\infty} \frac{n}{n^2+2} = \frac{\infty}{\infty}$$

Dividing both terms by n, we have

$$\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} \frac{1}{n+\frac{2}{n}}$$

We know that $\frac{2}{n} \to 0$.

Therefore, $\lim_{n\to+\infty} b_n = \left(\frac{1}{\infty+0}\right) = 0$

3. $c_n = \frac{5^n}{3^{2n}}$

$$\frac{5^n}{3^{2n}} = \left(\frac{5}{9}\right)^n$$

Because $-1 < \frac{5}{9} < 1$, $\lim_{n\to+\infty} c_n = 0$

Your turn!

Show that the following sequences tend to zero, using the previous properties to justify your answer.

1. $a_n = \frac{1}{n^2+1}$
2. $b_n = \frac{-2}{1+\sqrt{n}}$
3. $c_n = \frac{(-1)^n}{n^3}$

Your turn!

1. $n^2 \to +\infty$, $n^2 + 1 \to +\infty$, $\frac{1}{n^2+1} \to 0$
2. $\sqrt{n} \to +\infty$, $1 + \sqrt{n} \to +\infty$, $\frac{1}{1+\sqrt{n}} \to 0$

   $$-2 \cdot \frac{1}{1+\sqrt{n}} \to 0$$

3. $n^3 \to +\infty$, so $\frac{1}{n^3} \to 0$ and $\frac{(-1)^n}{n^3} \to 0$

So far, we have been dealing with limits of sequences, which, as you should recall, are functions whose domain is the natural numbers.

It is also possible to compute limits of other functions, at any point in the domain of the function. It does not have to be always at infinity!

Limit Laws

- When dealing with long or complicated limits, it can be useful to apply the following theorems:

1. Let a and k be real numbers.

   a) $\lim_{x \to a} k = k$

   b) $\lim_{x \to a} x = a$

   c) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

   d) $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

2. Let a be a real number, and suppose that

$$\lim_{x\to a} f(x) = L \quad and \quad \lim_{x\to a} g(x) = M$$

Then,

a) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$

b) $\lim_{x\to a}[f(x) \cdot g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L \cdot M$

c) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$, provided $M \neq 0$

- $$\lim_{x\to a} f(x) = L \quad and \lim_{x\to a} g(x) = M$$

d) $$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L},$$

provided $L > 0$ if n is even

e) $$\lim_{x\to a}[cf(x)] = c \cdot \lim_{x\to a} f(x) = c \cdot M$$

f) $$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n,$$

where n is a positive integer.

Your turn!

- Write in words the meaning of the following theorems:

1. $\lim_{x \to a} k = k$
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$

Your turn!

1. $\lim_{x \to a} k = k$

The limit of a constant is the constant itself.

2. $\lim_{x \to a} x = a$

The limit of x as x approaches a is a.

3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$

We can split the limit of a sum (difference) in two parts: the

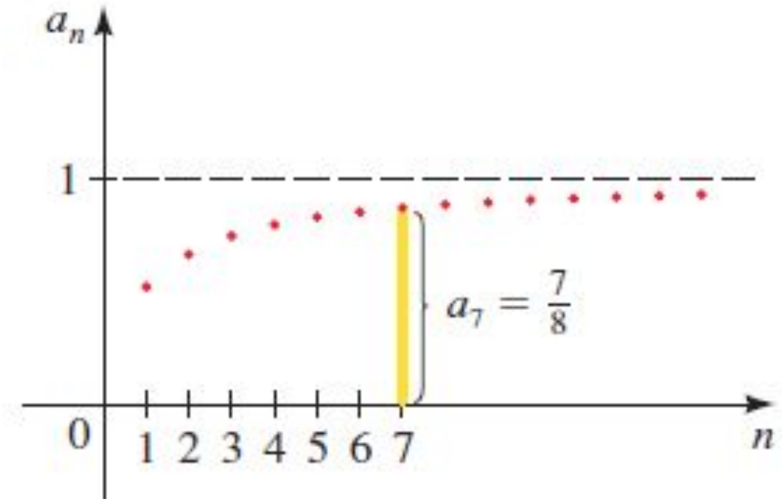
Convergent and divergent sequences

- A sequence is **convergent** when it tends to a real number.
- If a sequence tends to infinity or if it oscillates between two or more limits, then we say it is **divergent**.
- A sequence cannot have two different limits.

Example 3:

Plot the first 14 terms of the sequence $u_n = \frac{n}{n+1}$, and determine whether it is convergent or not.

What will be the difference between the graph of the sequence and the graph of the function $f(x) = \frac{x}{x+1}$?



Solution:

It's convergent.

It tends to 1.

Example 4: Determine whether the following sequences are convergent or divergent.

1. $s_n = \frac{1}{n} + 2$
2. $t_n = \frac{(-1)^n \cdot 2}{3^n}$
3. $u_n = \frac{4+n}{\sqrt{n}}$

Solution:

1. $s_n = \frac{1}{n} + 2$ is convergent.

$$\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} \frac{1}{n} + \lim_{n \to +\infty} 2 = 0 + 2 = 2$$

2. $t_n = \frac{(-1)^n \cdot 2}{3^n}$ is convergent.

$$\lim_{n \to +\infty} t_n = \lim_{n \to +\infty} (-1)^n \cdot \lim_{n \to +\infty} 2 \cdot \lim_{n \to +\infty} \frac{1}{3^n}$$
$$= \lim_{n \to +\infty} (-1)^n \cdot 2 \cdot 0 = 0$$

Why it isn't a problem that we don't know $\lim_{n \to +\infty} (-1)^n$?

3. $u_n = \frac{4+n}{\sqrt{n}}$ is divergent.

$$\lim_{n\to+\infty} \frac{4+n}{\sqrt{n}} = \lim_{n\to+\infty} \frac{4}{\sqrt{n}} + \lim_{n\to+\infty} \frac{n}{\sqrt{n}}$$

$$= 0 + \lim_{n\to+\infty} \sqrt{n} = +\infty$$

Why don't we need any restrictions for $\sqrt{\mathrm{n}}$?

Napier's Number, *e*

The mathematical constant **e** is a real, irrational and transcendental number approximately equal to:

e ≈ 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69995 95749 66967 62772 40766 30353 54759 45713 82178 53516 64274 27466 39193 20030 59921 81741 35966 29043 57290 03342 95260 59563 07381 32328 62794 34907 63233 82988 07531 95251 01901…

A special convergent sequence

- Compare the first digits of Napier's number with the value of the sequence

$$u_m = \left(1 + \frac{1}{m}\right)^m$$

e ≈ 2.71828 18284 …

In fact, $e = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m$

	A	B
1	m	(1+1/m)^m
2	1	2
3	10	2.59374246
4	100	2.704813829
5	1000	2.716923932
6	10000	2.718145927
7	100000	2.718268237
8	1000000	2.718280469
9	10000000	2.718281694
10	100000000	2.718281786
11	1000000000	2.718282031

Example 5:

Use

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

and the limit laws to compute the following limits:

1. $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+3}$
2. $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{3n}$
3. $\lim_{n\to+\infty}\left(1+\frac{1}{3^n}\right)^{3^n}$

Solution:

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

1. $$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+3}$$
$$= \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n \cdot \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^3 = e\cdot 1 = e$$

2. $$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{3n} = \lim_{n\to+\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^3 = e^3$$

- $$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

3. First note that if $n \to +\infty$, $3^n \to +\infty$. If we take $\mathrm{u} = 3^n$, then

$$\lim_{n \to +\infty} \left(1 + \frac{1}{3^n}\right)^{3^n} = \lim_{u \to +\infty} \left(1 + \frac{1}{u}\right)^u = e$$

Rules

- If $a_n \to +\infty$ and $b_n \to +\infty$, then
$$\lim_{n\to+\infty}(a_n + b_n) = +\infty$$
- If $a_n \to -\infty$ and $b_n \to -\infty$, then
$$\lim_{n\to+\infty}(a_n + b_n) = -\infty$$
- If $a_n \to +\infty$ and $b_n \to -\infty$, then
$$\lim_{n\to+\infty} a_n b_n = -\infty$$
- If $a_n \to a, a \in \mathbb{R}\backslash\{0\},$ and $b_n \to \pm\infty$, then
$$\lim_{n\to+\infty} \frac{a_n}{b_n} = 0$$

Your turn!

Fill in the blanks:

- If $a > 1$, then $a^{+\infty} = \square$ and $a^{-\infty} = \square = \square$
- If $0 < a < 1$, then $a^{+\infty} = \square$ and $a^{-\infty} = \square$
- $(+\infty)^{+\infty} = \square$
- $(+\infty)^{-\infty} = \square = \square$

Your turn!

Fill in the blanks:

- If $a > 1$, then $a^{+\infty} = \boxed{+\infty}$ and $a^{-\infty} = \boxed{\dfrac{1}{a^{+\infty}}} = \boxed{0}$
- If $0 < a < 1$, then $a^{+\infty} = \boxed{0}$ and $a^{-\infty} = \boxed{+\infty}$
- $(+\infty)^{+\infty} = \boxed{+\infty}$
- $(+\infty)^{-\infty} = \boxed{\dfrac{1}{+\infty^{+\infty}}} = \boxed{0}$

Indeterminations

As it happened in some of the previous examples, sometimes we can end up with **indeterminations** such as

$$\frac{0}{0} \quad \text{or} \quad 0^0$$

$$+\infty - \infty, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty^0 \quad \text{or} \quad 1^\infty$$

and we **cannot directly find the limit**.

To overcome this problem we need to rewrite the expression.

Example 6:

- Compute the limit $\lim\limits_{n \to +\infty} \dfrac{\frac{1}{n^2}}{\frac{1}{n}}$.

Solution:

Applying the limit laws,

$$\lim_{n \to +\infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = \frac{\lim\limits_{n \to +\infty} \frac{1}{n^2}}{\lim\limits_{n \to +\infty} \frac{1}{n}} = \frac{0}{0}$$

Indetermination!!!

Simplifying the expression,

$$\lim_{n \to +\infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = \lim_{n \to +\infty} \frac{n}{n^2} = \frac{+\infty}{+\infty}$$

Indetermination!!!

- However, simplifying the previous expression will allow us to avoid any indeterminations

$$\lim_{n\to+\infty}\frac{\frac{1}{n^2}}{\frac{1}{n}}=\lim_{n\to+\infty}\frac{n}{n^2}=\lim_{n\to+\infty}\frac{1}{n}=0$$

Example 7:

Compute, if possible, the limit at infinity of the sequence given by $t_n = \frac{n+3}{n+1}$

Solution: $$\lim_{n \to +\infty} \frac{n+3}{n+1}$$

If we simply replace the values of n, we will arrive at an indetermination of the type $\frac{+\infty}{+\infty}$.

Dividing the numerator by the denominator:

- Which means we can write

$$\frac{n+3}{n+1} = 1 + \frac{2}{n+1}$$

Now we can compute the limit

$$\lim_{n\to+\infty} \frac{n+3}{n+1} = \lim_{n\to+\infty} \left(1 + \frac{2}{n+1}\right)$$

$$= \lim_{n\to+\infty} 1 + \lim_{n\to+\infty} \frac{2}{n+1} = 1 + 0 = 1$$

Note: instead of dividing both polynomials, you could have noticed that

$$\frac{n+3}{n+1} = \frac{n+1+2}{n+1} = 1 + \frac{2}{n+1}$$

Example 8: Compute, if possible, the limit

$$\lim_{n\to+\infty}(-3n^2+n)$$

Solution: If we replace the values of n, we will arrive at an indetermination of the type $-\infty+\infty$. However, rewriting the expression so that $-3n^2$ is out of the brackets, we have:

$$\lim_{n\to+\infty}(-3n^2+n)=\lim_{n\to+\infty}-3n^2\left(1-\frac{1}{3n}\right)$$

$$=\lim_{n\to+\infty}-3n^2\lim_{n\to+\infty}\left(1-\frac{1}{3n}\right)=-\infty.1=-\infty$$

Example 9: As an example, let us examine the behaviour of

$$f(x) = x^2 - x + 1$$

for values of x close to 2.

x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

Simply **replacing x by numbers** in the expression, we see that when x gets closer and closer to 2, the value of $f(x)$ gets closer and closer to 3.

- We say that
“the limit of $f(x) = x^2 - x + 1$ is 3 as x approaches 2 **from either side**”
and we write it mathematically as

$$\lim_{x \to 2}(x^2 - x + 1) = 3$$

Note that in this example when we approach 2 from the left and from the right we get closer to the same value, 3. It doesn’t always have to be the case as we will see next lecture.

Your turn!

- Use numerical methods like in the previous exemple to make a conjecture about the behaviour of the function defined by

$$f(x) = x^2 - x + 2$$

for values of x near 2.

Your turn!

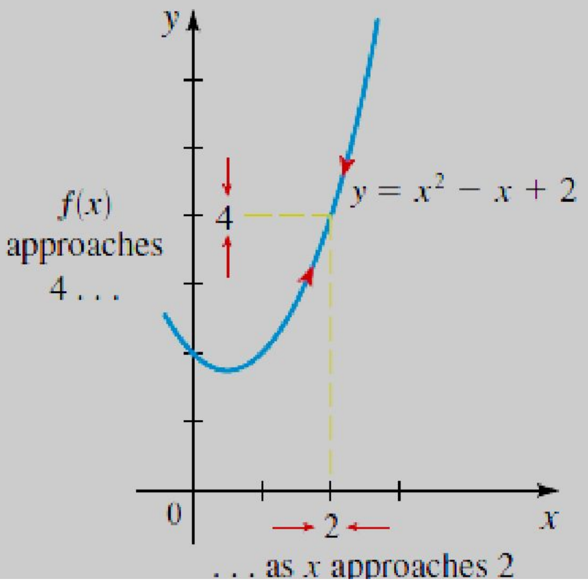
Use numerical methods like in the previous exemple to make a conjecture about the behaviour of the function defined by

$$f(x) = x^2 - x + 2$$

for values of x near 2.

x	$f(x)$
1.0	2.000000
1.5	2.750000
1.8	3.440000
1.9	3.710000
1.95	3.852500
1.99	3.970100
1.995	3.985025
1.999	3.997001

x	$f(x)$
3.0	8.000000
2.5	5.750000
2.2	4.640000
2.1	4.310000
2.05	4.152500
2.01	4.030100
2.005	4.015025
2.001	4.003001



So, how can we decide if a quantity is the limit value of another quantity? (Preview activity 4.1)

If the values of $f(x)$ can be made as close as we like to L taking values of x sufficiently close to a, but **not equal to *a***, then we write

$$\lim_{x \to a} f(x) = L \quad \text{or} \quad f(x) \to L \;\; as\; x \to a$$

which is read

"the limit of $f(x)$ as x approaches a is L".

Example 10:

The function $f(x) = \dfrac{x-1}{\sqrt{x}-1}$

is not defined for $x = 1$, or for negative x, but we can still try to see what is the behaviour of f for values of x closer and closer to 1.

x	0.99	0.999	0.9999	0.99999		1.00001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

Left side → ← Right side

- We can conjecture that $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = 2.$

Note that we cannot conclude **for sure** that the limit is 2, as by calculating the values with some numbers does not guarantee that the function will continue to behave this way.

x	0.99	0.999	0.9999	0.99999		1.00001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

Left side → ← Right side

Later we shall confirm that the limit is in fact 2.

Learning outcomes

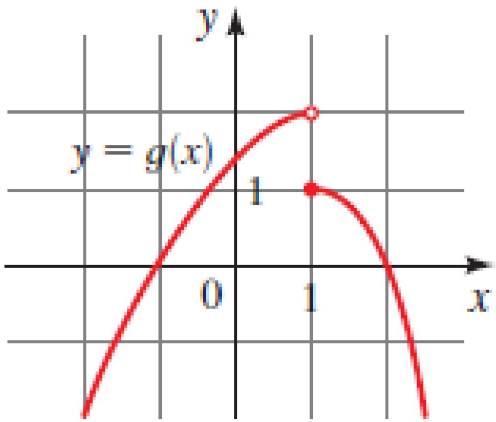
After this lecture, you should be able to

4.2.1 Compute the limit of sequences;

4.2.2 Determine whether a sequence is convergent or divergent;

4.2.3 Compute limits of other functions numerically and algebraically.

Preview activity: Limits



a function have two
ts at a point?

What if the function has a discontinuity (jump or hole)?

How would you change the definitions we have seen, or what would you add, to explain your answers?