Learning Objectives for Section 4.6

Matrix Equations and Systems of Linear Equations

- The student will be able to formulate matrix equations.
- The student will be able to use matrix equations to solve linear systems.
- The student will be able to solve applications using matrix equations.

Matrix Equations

- Let's review one property of solving equations involving real numbers. Recall

$$\text{If } ax = b \text{ then } x = \frac{1}{a}b \text{ , or } \frac{b}{a}$$

- A similar property of matrices will be used to solve systems of linear equations.

- Many of the basic properties of matrices are similar to the properties of real numbers, with the exception that matrix multiplication is not commutative.

Basic Properties of Matrices

Assuming that all products and sums are defined for the indicated matrices A, B, C, I, and 0, we have

- Addition Properties
  - Associative: $(A + B) + C = A + (B + C)$
  - Commutative: $A + B = B + A$
  - Additive Identity: $A + 0 = 0 + A = A$
  - Additive Inverse: $A + (-A) = (-A) + A = 0$

Basic Properties of Matrices (continued)

- Multiplication Properties
  - Associative Property: $A(BC) = (AB)C$
  - Multiplicative identity: $AI = IA = A$
  - Multiplicative inverse: If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$
- Combined Properties
  - Left distributive: $A(B + C) = AB + AC$
  - Right distributive: $(B + C)A = BA + CA$

Basic Properties of Matrices (continued)

- Equality
  - Addition: If $A = B$, then $A + C = B + C$
  - Left multiplication: If $A = B$, then $CA = CB$
  - Right multiplication: If $A = B$, then $AC = BC$

The use of these properties is best illustrated by an example of solving a matrix equation.

Example: Given an n x n matrix A and an n x p matrix B and a third matrix denoted by X, we will solve the matrix equation $AX = B$ for X.

Solving a Matrix Equation

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$(I_n)X = A^{-1}B$$

$$X = A^{-1}B$$

Reasons for each step:

1. Given; since A is n x n, X must by n x p.
2. Multiply on the left by A^{-1}.
3. Associative property of matrices
4. Property of matrix inverses.
5. Property of the identity matrix
6. **Solution**. Note A^{-1} is on the left of B. The order cannot be reversed because matrix multiplication is not commutative.

Example

- **Example:** Use matrix inverses to solve the system

$$\begin{aligned} x \quad &+ y &+2z &= 1 \\ 2x \quad &+ y & &= 2 \\ x \quad &+2y &+2z &= 3 \end{aligned}$$

Example

- **Example:** Use matrix inverses to solve the system

$$\begin{aligned} x \quad &+y \quad +2z = 1 \\ 2x \quad &+y \qquad\quad\ = 2 \\ x \quad &+2y \quad +2z = 3 \end{aligned}$$

- **Solution:**
  - Write out the matrix of coefficients A, the matrix X containing the variables x, y, and z, and the column matrix B containing the numbers on the right hand side of the equal sign.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example (continued)

- Form the matrix equation $AX = B$. Multiply the 3 x 3 matrix A by the 3 x 1 matrix X to verify that this multiplication produces the 3 x 3 system at the bottom:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{aligned} x \quad &+y \quad +2z &= 1 \\ 2x \quad &+y &= 2 \\ x \quad &+2y \quad +2z &= 3 \end{aligned}$$

Example (continued)

- If the matrix A^{-1} exists, then the solution is determined by multiplying A^{-1} by the matrix B. Since A^{-1} is 3 x 3 and B is 3 x 1, the resulting product will have dimensions 3 x 1 and will store the values of x, y and z.
- A^{-1} can be determined by the methods of a previous section or by using a computer or calculator. The resulting equation is shown at the right:

$$X = A^{-1}B$$

$$X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example Solution

The product of A^{-1} and B is

$$X = A^{-1}B$$

$$X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 \\ 2 \\ \frac{-1}{2} \end{bmatrix}$$

The solution can be read off from the X matrix:

$x = 0$,
$y = 2$,
$z = -1/2$

Written as an ordered triple of numbers, the solution is
(0, 2, -1/2)

Another Example

Example: Solve the system on the right using the inverse matrix method.

$$x + 2y + z = 1$$
$$2x - y + 2z = 2$$
$$3x + y + 3z = 4$$

Another Example

Example: Solve the system on the right using the inverse matrix method.

$$\begin{aligned} x + 2y + z &= 1 \\ 2x - y + 2z &= 2 \\ 3x + y + 3z &= 4 \end{aligned}$$

Solution:

The coefficient matrix A is displayed at the right. The inverse of A does not exist. (We can determine this by using a calculator.) We cannot use the inverse matrix method. Whenever the inverse of a matrix does not exist, we say that the matrix is **singular.**

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

Cases When Matrix Techniques Do Not Work

- There are two cases when inverse methods will not work:

  1. If the coefficient matrix is singular

  2. If the number of variables is not the same as the number of equations.

Application

- **Production scheduling:** Labor and material costs for manufacturing two guitar models are given in the table below: Suppose that in a given week $1800 is used for labor and $1200 used for materials. How many of each model should be produced to use exactly each of these allocations?

Guitar model	Labor cost	Material cost
A	$30	$20
B	$40	$30

Solution

Let x be the number of model A guitars to produce and y represent the number of model B guitars. Then, multiplying the labor costs for each guitar by the number of guitars produced, we have

$30x + 40y = 1800$

Since the material costs are \$20 and \$30 for models A and B respectively, we have

$20x + 30y = 1200.$

This gives us the system of linear equations:

$30x + 40y = 1800$

$20x + 30y = 1200$

We can write this as a matrix equation:

$$\begin{bmatrix} 30 & 40 \\ 20 & 30 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1800 \\ 1200 \end{bmatrix}$$

Solution (continued)

$$X = A^{-1}B$$

$$A = \begin{bmatrix} 30 & 40 \\ 20 & 30 \end{bmatrix}$$

The inverse of matrix A is $\begin{bmatrix} 0.3 & -0.4 \\ -0.2 & 0.3 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.3 & -0.4 \\ -0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 1800 \\ 1200 \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \end{bmatrix}$$

- **Solution:**
  Produce 60 model A guitars and no model B guitars.