Lecture # _3

Elements of probability

Definition 1

A simple event is an outcome of an experiment that can not be decomposed into a simpler outcome

Example 1. Tossing 1 fair coin

$\{\omega_1, \omega_2\}$ $\qquad \omega_1 \to$ head (H)

$\omega_2 \to$ tail (T)

Example 2. Tossing 3 fair coins at once

$$\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}$$

$\omega_1 \to \text{HHH}$ $\omega_5 \to \text{TTT}$

$\omega_2 \to \text{TTH}$ $\omega_6 \to \text{HHT}$

$\omega_3 \to \text{THT}$ $\omega_7 \to \text{HTH}$

$\omega_4 \to \text{HTT}$ $\omega_8 \to \text{THH}$

A random variable $X(\omega)$ is a function of ω, e.g., $X(\omega)$ is the number of heads in one trial, hence,

$$X(\omega_1) = 3, \quad X(\omega_6) = X(\omega_7) = X(\omega_8) = 2,$$
$$X(\omega_2) = X(\omega_3) = X(\omega_4) = 1, \quad X(\omega_5) = 0$$

Definition 2

An **event** is a collection of one or more simple events

Example 3.

$A - 2$ Heads in one toss

$A = \{\omega_6, \omega_7, \omega_8\}$

$B -$ More than one H in a trial

$B = \{\omega_1, \omega_6, \omega_7, \omega_8\}$

$C -$ Exactly 2 tails (T)

$C = \{\omega_2, \omega_3, \omega_4\}$

$D -$ At least one H or one T

$D = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}$

Definition 3

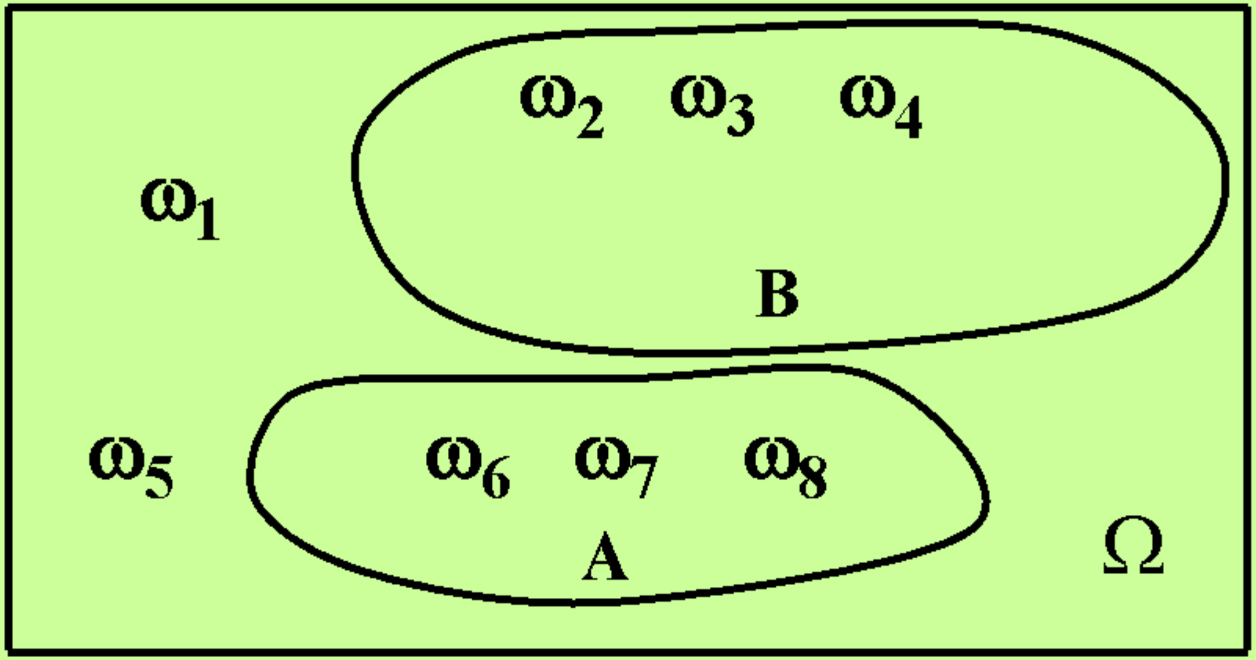
The sample space of an experiment is the collection of all possible simple events

Denote it by Ω

For example 1: $\Omega = \{\omega_1, \omega_2\}$

For example 2 : $\Omega = \{\omega_1, \omega_2, ..., \omega_8\}$

Venn diagram



Definition 4.

The probability p of a simple event is a number that measures the likelihood that the event will occur when the experiment is performed.

I. Classical definition of probabilities. If one has n equiprobable simple events $\omega_1, ..., \omega_n$, then the probability $p(A)$ of $A = \{\omega_{i_1}, \omega_{i_2}, ..., \omega_{i_m}\}$ containing m simple events is

$$P(A) = \frac{m}{n}$$

Evidently, that $P(\Omega) = 1$.

II. We may take p being equal to the relative frequency f_i/n of a simple ith event if n is very large.

III. We may select p based on a priori knowledge of a situation under study.

The case III is more frequently occurred. We guess or formulate a hypothesis H concerning p and, using statistics, test this hypothesis based on realizations $(x_1, ..., x_n)$ of a sample $X_n = \{X_1, ..., X_n\}$.

Definition 5

The probability P(A) of an event A is calculated by summing the probabilities of the simple events belonging to A.

Steps for calculating probabilities

- 1. List the simple events
- 2. Assign probabilities to simple events
- 3. Determine the number of simple events containing in the event
- 4. Sum the simple event probabilities to obtain the event probability

For example 2

$$p(\omega_i) = 1/8, \quad i = \overline{1,8}$$

$$p(A) = p(2H) = p\{\omega_6, \omega_7, \omega_8\} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$p(B) = P(\text{more than } 1H) =$$

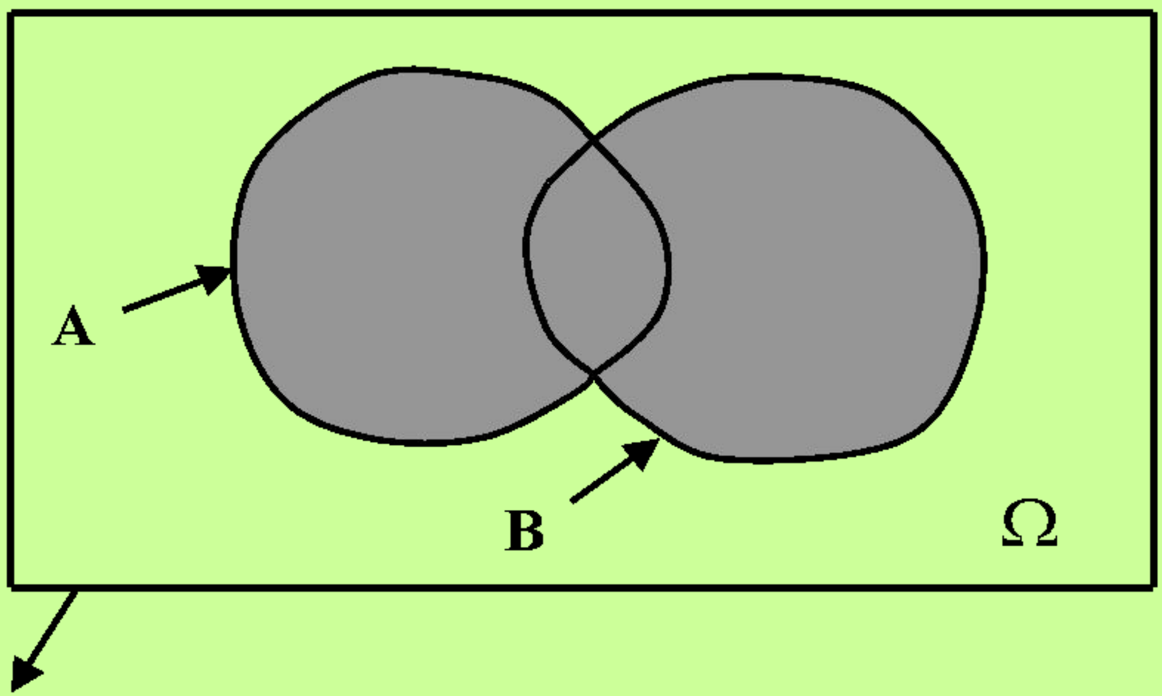
$$= p\{\omega_1, \omega_6, \omega_7, \omega_8\} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$p(D) = p(\Omega) = 1 \text{ (certain event)}$$

Compound events

Definition 6

The union of A & B is the event that occurs if either A or B or both occur $(A \cup B)$



$\omega_i, i=\overline{1,n},$ belong A or B or to both of them

Definition 7

The **intersection** of A & B is the event that occurs if **both** A **and** B occur $(A \cap B)$

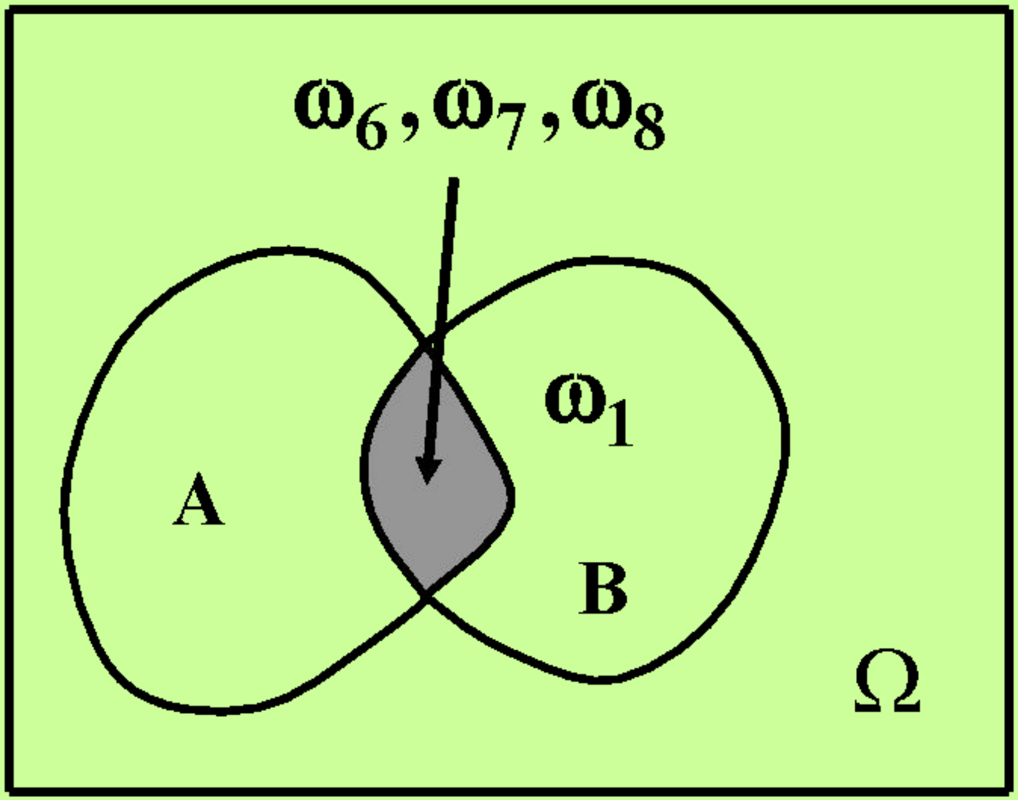
ω_i, belong to both A and B

$\overline{A}$ - the **complement** event to A

ω_i of $\overline{A}$ do not belong to A!

Example 3

$A = \{\omega_6, \omega_7, \omega_8\}, \quad B = \{\omega_1, \omega_6, \omega_7, \omega_8\}$



$A \cap B = \{\omega_6, \omega_7, \omega_8\}$

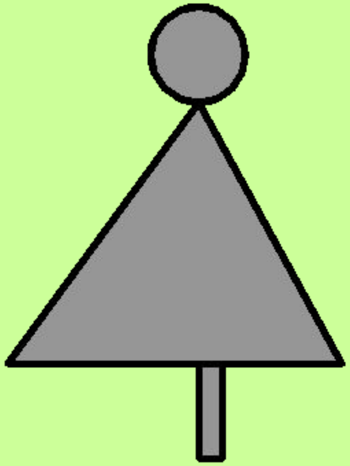
$A \cup B = \{\omega_1, \omega_6, \omega_7, \omega_8\}$

Let a set $\Omega = \{\omega_1, ..., \omega_n\}$ be given, then all possible events (combinations of ω_i th) $+ \Omega + \varnothing$ (empty set, corresponding to the improbable event) is <u>algebra S of events</u>

If two events A & B are mutually exclusive (incompatible), then

$$P(A \cup B) = P(A) + P(B).$$

Suppose we have a target. A shooter produced a shot.



$A \Rightarrow$ to hit the target

$B \Rightarrow$ fail to hit

A&B are incompatible

$P(A) + P(B) = 1$

$A \cup B$ is the certain event

Axioms of probability

For every Ω (a set of all possible simple events) and S (algebra of events) we postulate

1) **Axiom 1.** For every $A \in S$ $P(A) \geq 0$

2) **Axiom 2.** $P(\Omega) = 1$ Ω-is certain event

3) **Axiom 3.** If $A \in S$ and $B \in S$ are incompatible, then

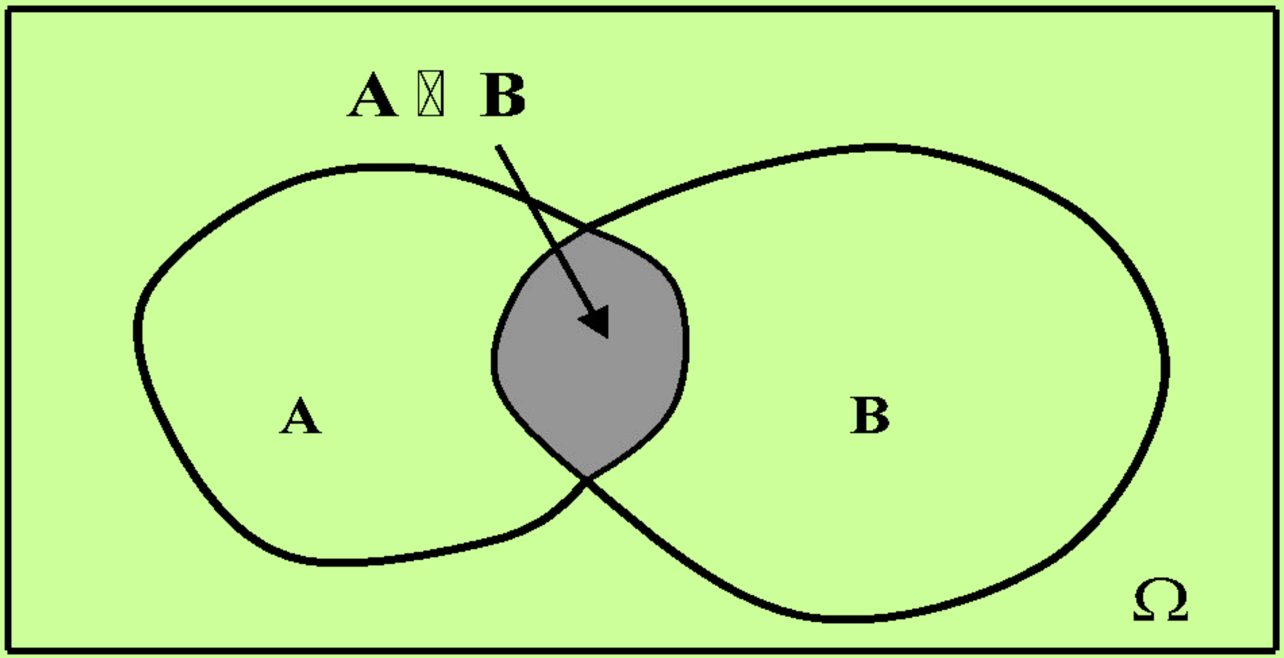
$$P(A \cup B) = P(A) + P(B).$$

The following two rules are consequences of axioms:

Additive rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

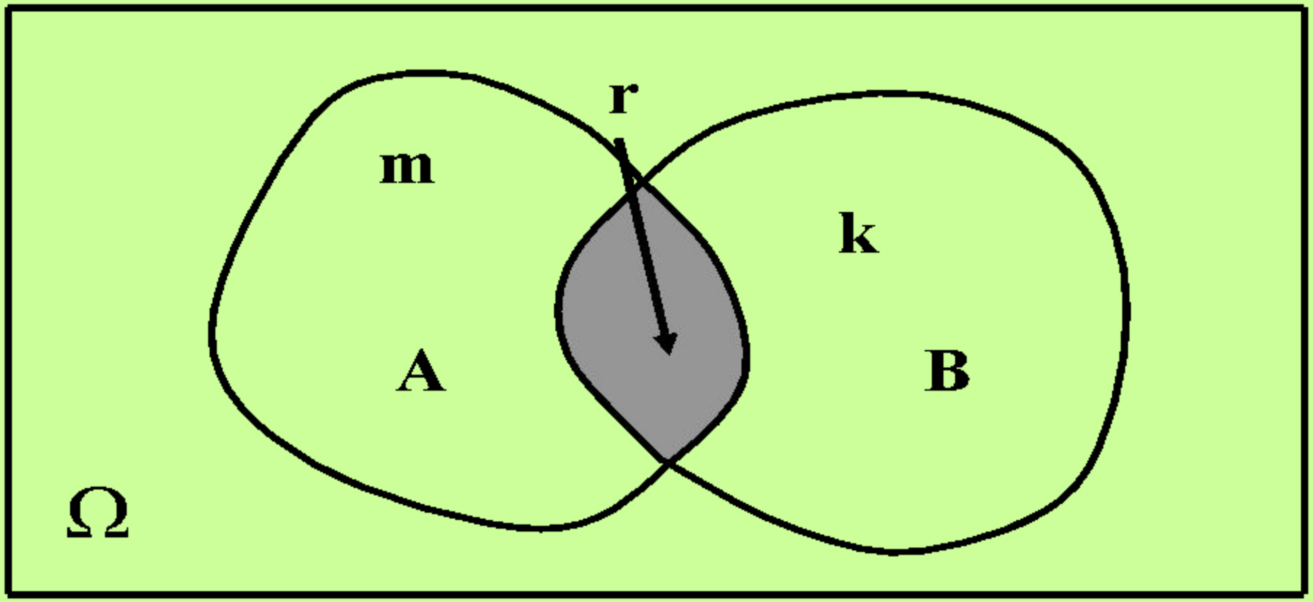
Multiplicative rule



$$P(A \cap B) = P(A \mid B)P(B)$$

$$P(A \cap B) = P(B \mid A)P(A)$$

The conditional probability



A contains m elements of Ω

B contains k elements of Ω

$A \cap B$ contains r elements of Ω

$\Omega \in \{\omega_1, \ldots, \omega_n\}$, ω_ith are equiprobable

$$P(A \cap B) = \frac{r}{n} \qquad P(B) = \frac{k}{n}$$

Let the event B occurred. What is the prob. for A to occur?

$$P(A \mid B) = \frac{r}{k} = \frac{r/n}{k/n} = \frac{P(A \cap B)}{P(B)}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Independence

Events A and B are independent if the assumption that B has occurred does not alter the probability that A occurs, or if

$$P(A \mid B) = P(A)$$

or

$$P(B \mid A) = P(B)$$

Otherwise, A&B are dependent

Example 2

$$A = \{\omega_6, \omega_7, \omega_8\} \Rightarrow \text{two H in one tossing}$$

$$B = \{\omega_1, \omega_6, \omega_7, \omega_8\} \Rightarrow > \text{than 1 H in one tossing}$$

$$P(A) = \frac{3}{8} \quad P(B) = \frac{4}{8} \quad P(\Omega) = \frac{8}{8} = 1$$

$$A \cap B = \{\omega_6, \omega_7, \omega_8\} = A$$

$$P(A \cap B) = P(A) = \frac{3}{8}$$

$$P(A \cap B) = P(A \mid B)P(B) = \frac{3}{4} \cdot \frac{4}{8} = \frac{3}{8}$$

$$P(A) \cdot P(B) = \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16} \neq P(A \cap B) = \frac{3}{8}$$

Hence, A and B are <u>dependent</u>

$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)} = \frac{3/8}{1/2} = \frac{3}{4} \neq P(A) = \frac{3}{8}!!$$

For independent events

$$P(A \cap B) = P(A) \bullet P(B)$$

Random variable $X(\omega)$ (discrete)

$X(\omega)$ is the number of H for example 2. From the symmetry of a coin we may suppose that the probability p to obtain a head in one trial is $p = 1/2.(q = 1/2 = 1 - p)$. Then

$$P(X = k) = \binom{3}{k} p^k q^{3-k}, \qquad k = 0,1,2,3.$$

$$P(A) = P(X = 2) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{3-2} = 3 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

$$P(B) = P(X > 1) = P(X = 2) + P(X = 3) =$$

$$= \frac{3}{8} + \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

Events (X=2) & (X=3) are mutually exclusive, hence

$$P((X=2) \cup (X=3)) = P(X=2) + P(X=3).$$

We may consider P(X=k) as both known and known within the unknown parameter p.

Population moments (discrete case)

Let P (X = k) be the p. d. of X, k∈{set of discrete numbers}=K.

$$\sum_{k \in K} P(X = k) = 1$$

Definition 9

The initial population moment of P (X = k) of order m is

$$\alpha_m = \sum_{k \in K} k^m P(X = k) = E(X^m)$$

EX^m - expectation of X^m or

expected value of X^m

Definition 10

The central population moment of P (X = k) of order m is

$$\beta_m = \sum_{k \in K} (k - EX)^m P(X = k) = E(X - EX)^m$$

Definition 11 (p.210)

The mean or expected value of a discrete r. v. X is

$$\mu = EX = \sum_{k \in K} k P(X = k)$$

Definition 12

The variance of a discrete random variable X is

$$\sigma^2 = E(X-\mu)^2 = \sum_{k\in K}(k-\mu)^2 P(X=k)$$

Example Binomial distribution

$$P=(X=k)=\binom{n}{k}p^k(1-p)^{n-k}, \quad k=\overline{0,n} \Rightarrow$$

$$\Rightarrow K=\{0,1,\ldots,n\}$$

$$\mu = EX = \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} =$$

$$= \sum_{k=1}^{n} \frac{kn!}{k!(n-k)!}p^k(1-p)^{n-k} =$$

$$= \sum_{k=1}^{n} \frac{n(n-1)!}{(k-1)!(n-k)!}p \bullet p^{k-1}q^{n-k} =$$

$$= np\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k-1}q^{n-k} =$$

$$k-1 \to r \Rightarrow k = r+1$$

$$= np\sum_{r=0}^{n-1}\binom{n-1}{r}p^r q^{(n-1)-r} = np,$$

$$\sum_{r=0}^{n-1}\binom{n-1}{r}p^r q^{(n-1)-r} = (p+q)^{n-1} = 1$$

Thus,

$$\mu = EX = np$$

By the same lines

$$\sigma^2 = E(X-\mu)^2 = npq$$

Standard deviation

$$\sigma = \sqrt{npq} \quad (1a)$$