Hash Tables

SDP-4

Dictionary

- **Dictionary:**
  - Dynamic-set data structure for storing items indexed using *keys*.
  - Supports operations Insert, Search, and Delete.
  - Applications:
    - Symbol table of a compiler.
    - Memory-management tables in operating systems.
    - Large-scale distributed systems.
- **Hash Tables:**
  - Effective way of implementing dictionaries.
  - Generalization of ordinary arrays.

Direct-address Tables

- Direct-address Tables are ordinary arrays.
- Facilitate direct addressing.
  - Element whose key is k is obtained by indexing into the k^{th} position of the array.
- Applicable when we can afford to allocate an array with one position for every possible key.
  - i.e. when the universe of keys U is small.
- Dictionary operations can be implemented to take $O(1)$ time.
  - Details in Sec. 11.1.

Hash Tables

- **Notation:**
  - U – Universe of all possible keys.
  - K – Set of keys actually stored in the dictionary.
  - $|K| = n$.
- When U is very large,
  - Arrays are not practical.
  - $|K| << |U|$.
- Use a table of size proportional to $|K|$ – The hash tables.
  - However, we lose the direct-addressing ability.
  - Define functions that map keys to slots of the hash table.

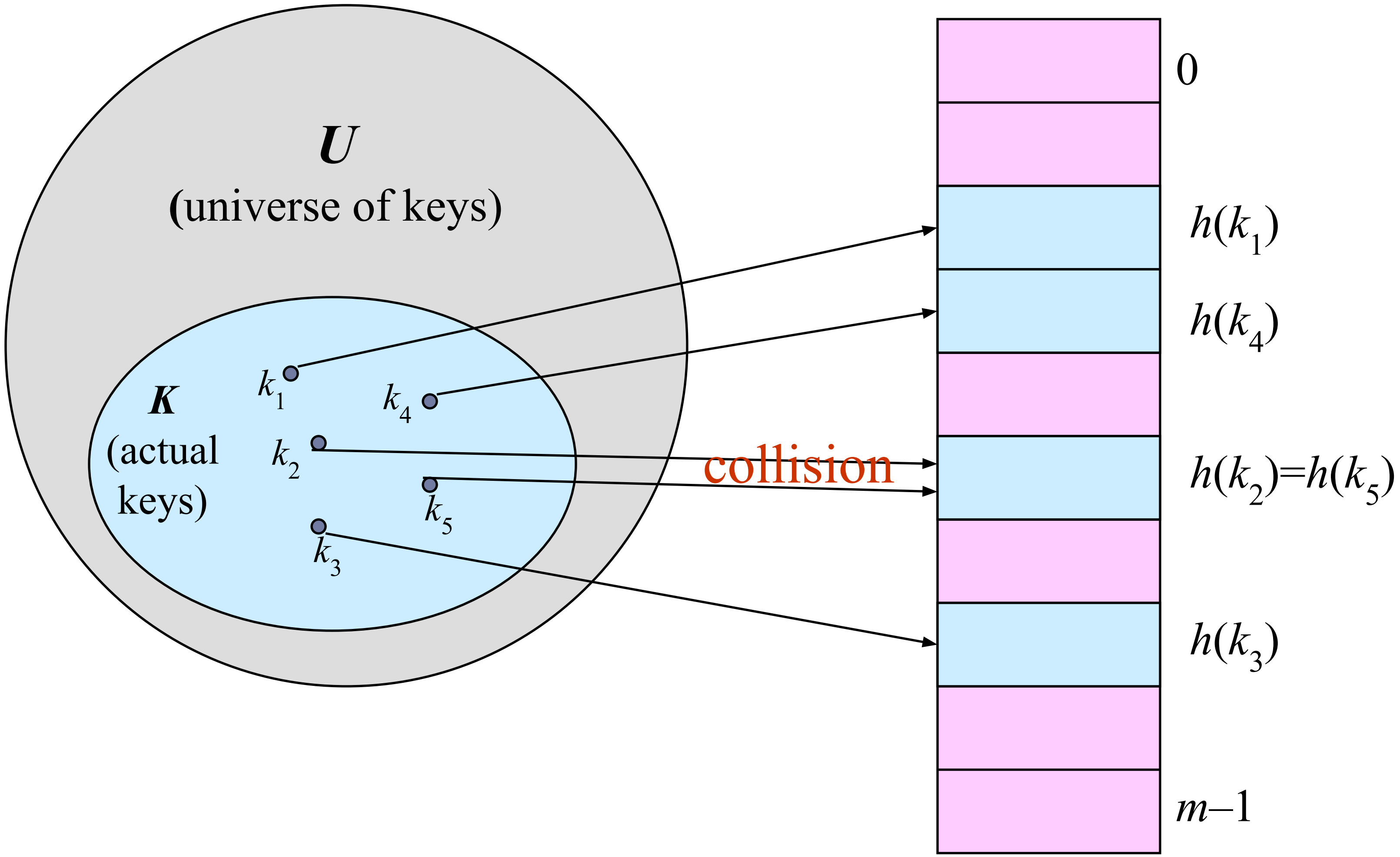
Hashing

- Hash function h: Mapping from U to the slots of a hash table $T[0..m-1]$.

$$h : U \longrightarrow \{0, 1, \ldots, m-1\}$$

- With arrays, key k maps to slot $A[k]$.
- With hash tables, key k maps or "hashes" to slot $T[h[k]]$.
- $h[k]$ is the *hash value* of key k.

Hashing

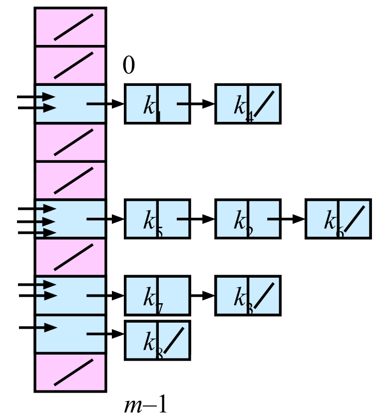

U
(universe of keys)
K
(actual
keys)
k1
k4
k2
k5
k3
collision
0
h(k1)
h(k4)
h(k2)=h(k5)
h(k3)
m–1

Issues with Hashing

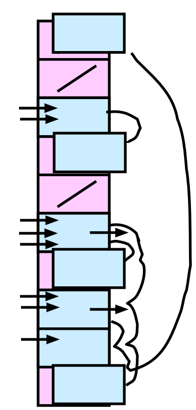
- Multiple keys can hash to the same slot – collisions are possible.
  - Design hash functions such that collisions are minimized.
  - But avoiding collisions is impossible.
    - Design collision-resolution techniques.
- Search will cost $\Theta(n)$ time in the worst case.
  - However, all operations can be made to have an expected complexity of $\Theta(1)$.

Methods of Resolution

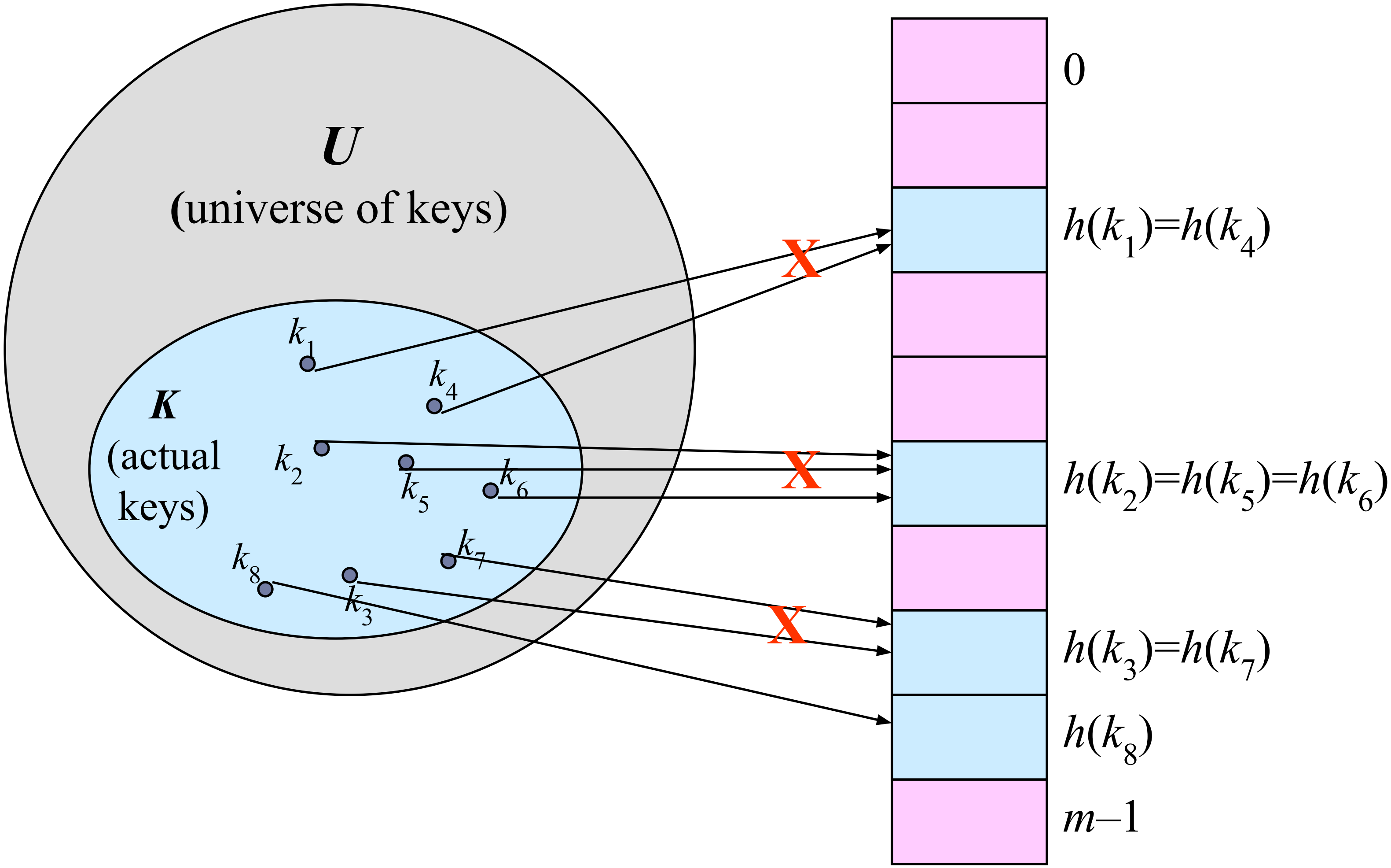
- Chaining:
  - Store all elements that hash to the same slot in a linked list.
  - Store a pointer to the head of the linked list in the hash table slot.



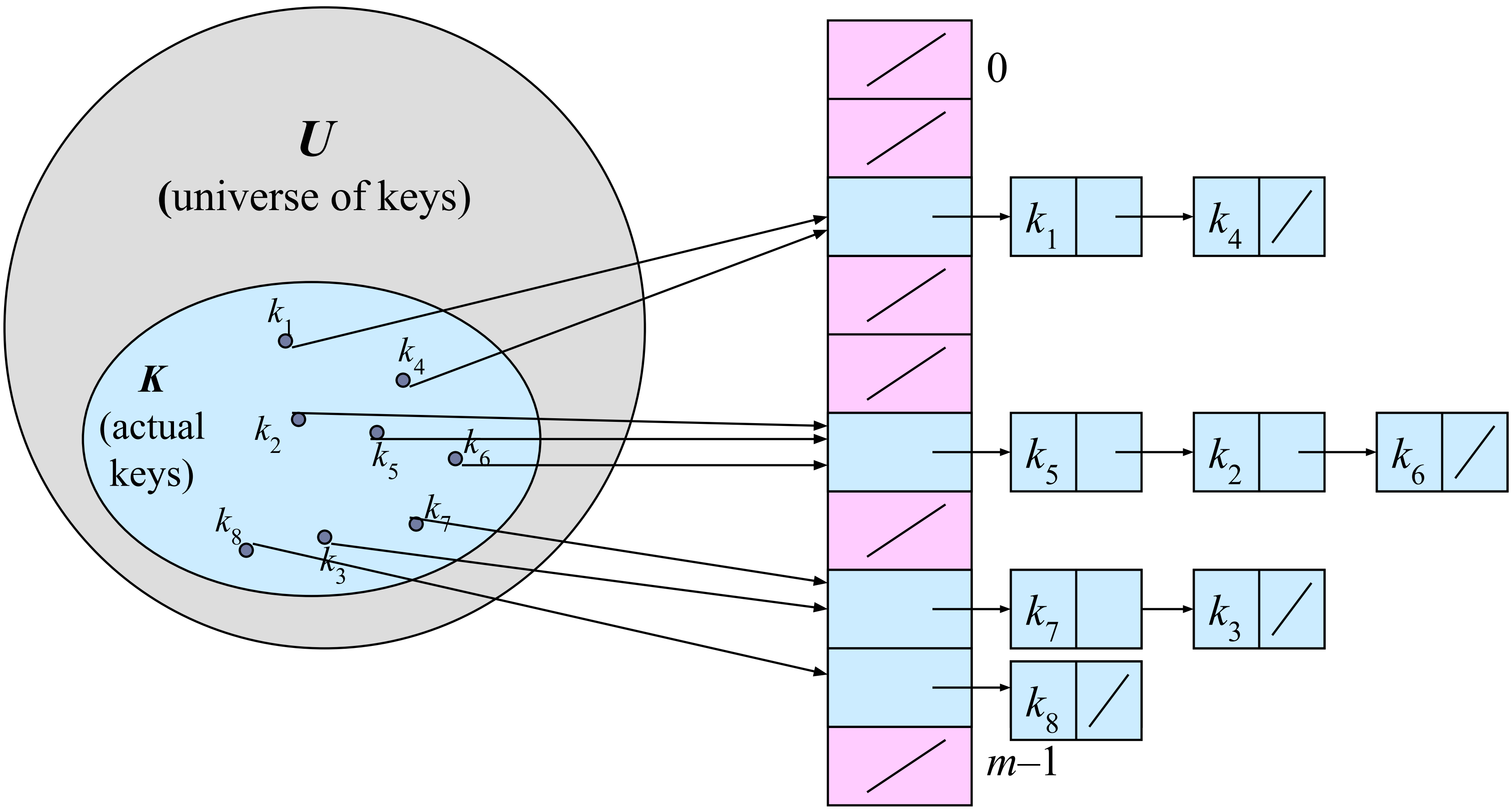
- Open Addressing:
  - All elements stored in hash table itself.
  - When collisions occur, use a systematic (consistent) procedure to store elements in free slots of the table.



Collision Resolution by Chaining



Collision Resolution by Chaining



Hashing with Chaining

Dictionary Operations:

- Chained-Hash-Insert (T, x)
  - Insert x at the head of list $T[h(key[x])]$.
  - Worst-case complexity – $O(1)$.
- Chained-Hash-Delete (T, x)
  - Delete x from the list $T[h(key[x])]$.
  - Worst-case complexity – proportional to length of list with singly-linked lists. $O(1)$ with doubly-linked lists.
- Chained-Hash-Search (T, k)
  - Search an element with key k in list $T[h(k)]$.
  - Worst-case complexity – proportional to length of list.

Analysis on Chained-Hash-Search

- Load factor $\alpha = n/m$ = average keys per slot.
  - m – number of slots.
  - n – number of elements stored in the hash table.
- Worst-case complexity: $\Theta(n)$ + time to compute $h(k)$.

- Average depends on how h distributes keys among m slots.
- **Assume**
  - *Simple uniform hashing*.
    - Any key is equally likely to hash into any of the m slots, independent of where any other key hashes to.
  - $O(1)$ time to compute $h(k)$.
- Time to search for an element with key k is $\Theta(|T[h(k)]|)$.
- Expected length of a linked list = load factor = $\alpha = n/m$.

Expected Cost of an Unsuccessful Search

Theorem:
An unsuccessful search takes expected time $\Theta(1+\alpha)$.

Proof:

- Any key not already in the table is equally likely to hash to any of the *m* slots.
- To search unsuccessfully for any key *k*, need to search to the end of the list $T[h(k)]$, whose expected length is α.
- Adding the time to compute the hash function, the total time required is $\Theta(1+\alpha)$.

Expected Cost of a Successful Search

Theorem:
A successful search takes expected time $\Theta(1+\alpha)$.

Proof:

- The probability that a list is searched is proportional to the number of elements it contains.
- Assume that the element being searched for is equally likely to be any of the *n* elements in the table.
- The number of elements examined during a successful search for an element *x* is 1 more than the number of elements that appear before *x* in *x*'s list.
  - These are the elements inserted ***after x*** was inserted.
- Goal:
  - Find the average, over the *n* elements *x* in the table, of how many elements were inserted into *x*'s list after *x* was inserted.

Expected Cost of a Successful Search

Theorem:
A successful search takes expected time $\Theta(1+\alpha)$.

Proof (contd):

- Let x_i be the i^{th} element inserted into the table, and let $k_i = key[x_i]$.
- Define indicator random variables $X_{ij} = I\{h(k_i) = h(k_j)\}$, for all i, j.
- Simple uniform hashing $\Rightarrow \Pr\{h(k_i) = h(k_j)\} = 1/m$

  $\Rightarrow E[X_{ij}] = 1/m$.
- Expected number of elements examined in a successful search is:

$$E\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}X_{ij}\right)\right]$$

No. of elements inserted after x_i into the same slot as x_i.

Proof – Contd.

$$E\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}X_{ij}\right)\right]$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}E[X_{ij}]\right)$$

(linearity of expectation)

$$=\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right)$$

$$=1+\frac{1}{nm}\sum_{i=1}^{n}(n-i)$$

$$=1+\frac{1}{nm}\left(\sum_{i=1}^{n}n-\sum_{i=1}^{n}i\right)$$

$$=1+\frac{1}{nm}\left(n^2-\frac{n(n+1)}{2}\right)$$

$$=1+\frac{n-1}{2m}$$

$$=1+\frac{\alpha}{2}-\frac{\alpha}{2n}$$

Expected total time for a successful search = Time to compute hash function + Time to search = $O(2+\alpha/2 - \alpha/2n) = O(1+\alpha)$.

Expected Cost – Interpretation

- If $n = O(m)$, then $\alpha = n/m = O(m)/m = O(1)$.
  $\Rightarrow$ Searching takes constant time on average.
- Insertion is $O(1)$ in the worst case.
- Deletion takes $O(1)$ worst-case time when lists are doubly linked.
- Hence, all dictionary operations take $O(1)$ time on average with hash tables with chaining.

Good Hash Functions

- Satisfy the assumption of *simple uniform hashing*.
  - Not possible to satisfy the assumption in practice.
- Often use heuristics, based on the domain of the keys, to create a hash function that performs well.
- Regularity in key distribution should not affect uniformity. Hash value should be independent of any patterns that might exist in the data.
  - E.g. Each key is drawn independently from U according to a probability distribution P:

$$\sum_{k:h(k)=j} P(k) = 1/m \quad \text{for } j = 0, 1, \ldots, m-1.$$

  - An example is the division method.

Keys as Natural Numbers

- Hash functions assume that the keys are natural numbers.
- When they are not, have to interpret them as natural numbers.
- Example: Interpret a character string as an integer expressed in some radix notation. Suppose the string is CLRS:
  - ASCII values: C=67, L=76, R=82, S=83.
  - There are 128 basic ASCII values.
  - So, CLRS = $67 \cdot 128^3 + 76 \cdot 128^2 + 82 \cdot 128^1 + 83 \cdot 128^0$ $= 141{,}764{,}947$.

Division Method

- Map a key k into one of the m slots by taking the remainder of k divided by m. That is,

$$h(k) = k \bmod m$$

- Example: $m = 31$ and $k = 78 \Rightarrow h(k) = 16$.
- **Advantage:** Fast, since requires just one division operation.
- **Disadvantage:** Have to avoid certain values of m.
  - Don't pick certain values, such as $m=2^p$
  - Or hash won't depend on all bits of k.
- **Good choice for m:**
  - Primes, not too close to power of 2 (or 10) are good.

Multiplication Method

- If $0 < A < 1$, $h(k) = \lfloor m\,(kA \bmod 1) \rfloor = \lfloor m\,(kA - \lfloor kA \rfloor)\rfloor$ where $kA \bmod 1$ means the fractional part of kA, *i.e.*, $kA - \lfloor kA \rfloor$.
- Disadvantage: Slower than the division method.
- Advantage: Value of m is not critical.
  - Typically chosen as a power of 2, i.e., $m = 2^p$, which makes implementation easy.

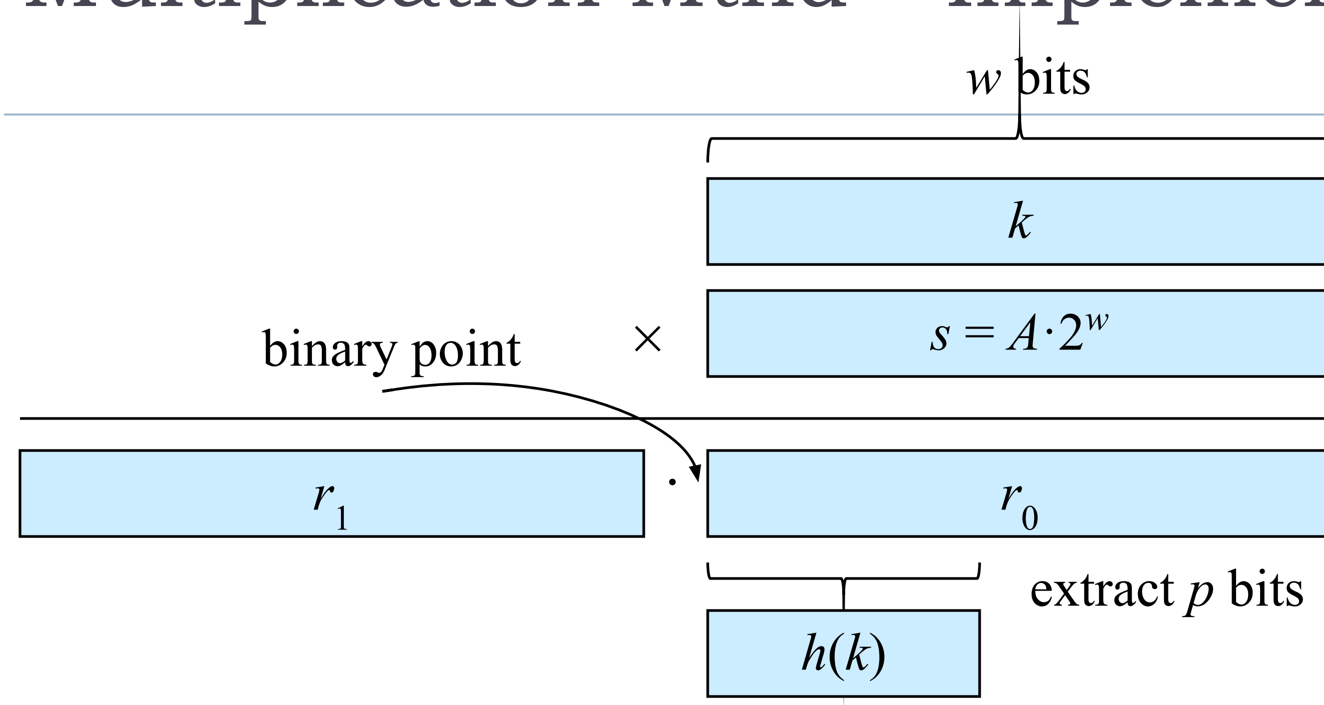
- Example: $m = 1000$, $k = 123$, $A \approx 0.6180339887\ldots$

$$h(k) = \lfloor 1000(123 \cdot 0.6180339887 \bmod 1)\rfloor$$
$$= \lfloor 1000 \cdot 0.018169\ldots \rfloor = 18.$$

Multiplication Mthd. – Implementation

- Choose $m = 2^p$, for some integer p.
- Let the word size of the machine be w bits.
- Assume that k fits into a single word. (k takes w bits.)
- Let $0 < s < 2^w$. (s takes w bits.)
- Restrict A to be of the form $s/2^w$.
- Let $k \times s = r_1 \cdot 2^w + r_0$.
- r_1 holds the integer part of kA ($\lfloor kA \rfloor$) and r_0 holds the fractional part of kA ($kA \bmod 1 = kA - \lfloor kA \rfloor$).
- We don't care about the integer part of kA.
  - So, just use r_0, and forget about r_1.

Multiplication Mthd – Implementation



- We want $\lfloor m\ (kA \bmod 1) \rfloor$. We could get that by shifting r_0 to the left by $p = \lg m$ bits and then taking the p bits that were shifted to the left of the binary point.
- But, we don't need to shift. Just take the p most significant bits of r_0.

How to choose *A*?

- Another example: On board.
- How to choose *A*?
  - The multiplication method works with any legal value of *A*.
  - But it works better with some values than with others, depending on the keys being hashed.
  - Knuth suggests using $A \approx (\sqrt{5} - 1)/2$.