Лекция 5.
Математические методы физики волновых явлений – теория

1. Общие уравнения для вектора состояния физической системы, построение дисперсионной функции

2. Задача с начальными условиями. Метод собственных волн

2. Характеристическая функция вектора состояния. Дисперсионный оператор

4. Метод преобразования Лапласа

$$\frac{\partial \psi_s}{\partial t} + \sum_{j=1}^{n} \left(A_{sj} \frac{\partial \psi_j}{\partial z} + B_{sj} \psi_j \right) = 0, \quad s = 1,2,\ldots,n$$

$$\frac{\partial E_x}{\partial t} + \frac{c}{\varepsilon_0} \frac{\partial B_y}{\partial z} = 0$$

$$\frac{\partial B_y}{\partial t} + c \frac{\partial E_x}{\partial z} = 0$$

$$\left\{ \psi_1(t,z), \psi_2(t,z), \ldots, \psi_n(t,z) \right\} \equiv \mathbf{\Psi}(t,z)$$

$$\mathbf{\Psi}(t,z) = \mathbf{\Phi}(\omega,k) \exp(-i\omega t + ikz)$$

,

$$\mathbf{\Phi}(\omega,k) = \left\{ \phi_1(\omega,k), \phi_2(\omega,k), \ldots, \phi_n(\omega,k) \right\}$$

$$\mathbf{\Psi}(t,z) = \left\{ E_x(t,z), B_y(t,z) \right\} = \\ = \left\{ e_x(\omega,k),\ b_y(\omega,k) \right\} \exp(-i\omega t + ikz)$$

$$-i\omega\, \phi_s(\omega,k) + \sum_{j=1}^{n} \left(ikA_{sj} + B_{sj} \right) \phi_j(\omega,k) = 0, \quad s = 1,2,\ldots,n$$

$$-i\omega\, e_x + i\left(c/\varepsilon_0\right) k\, b_y = 0$$

$$ick\, e_x - i\omega\, b_y = 0$$

$$\boxed{D(\omega,k) \equiv \det\left(-i\omega\delta_{sj} + ikA_{sj} + B_{sj} \right) = 0, \quad s,j = 1,2,\ldots,n}$$

$$D(\omega,k) \equiv -\omega^2 + k^2 c_0^2 = 0$$

$$\omega = \omega_m(k) \quad m = 1,2,\ldots,n \qquad V_{\Phi}^{(m)} = \frac{\omega_m(k)}{k}$$

$$\omega_1 = kc_0, \quad \omega_2 = -kc_0$$

$$\mathbf{\Psi}(t,z)=\sum_{m=1}^{n}\mathbf{\Psi}_m(t,z)=\sum_{m=1}^{n}A_m\vec{\Phi}_m(k)\exp\left[-i\omega_m(k)t+ikz\right]$$

$$\mathbf{\Phi}_m(\omega,k)=\mathbf{\Phi}\left(\omega_m(k),k\right)\equiv\mathbf{\Phi}_m(k)=\left\{\phi_1^{(m)}(k),\phi_2^{(m)}(k),\ldots,\phi_n^{(m)}(k)\right\}$$

$$\vec{\Psi}(t,z)=\left\{E_x,B_y\right\}=A_1\{e_x^{(1)},b_y^{(1)}\}\exp\left[ik(z-c_0t)\right]+A_2\{e_x^{(2)},b_y^{(2)}\}\exp\left[ik(z+c_0t)\right]$$

$$A_m\phi_1^{(m)}(k)\equiv A_m(k)\qquad \phi_s^{(m)}(k)=L_s\left(\omega_m(k),k\right)\cdot\phi_1^{(m)}(k),\qquad s=2,3,\ldots,n$$

$$\mathbf{\Psi}(t,z)=\left\{E_x,B_y\right\}=A_1\{1,\sqrt{\varepsilon_0}\}\exp\left[ik(z-c_0t)\right]+A_2\{1,-\sqrt{\varepsilon_0}\}\exp\left[ik(z+c_0t)\right]$$

$$\omega_m\left(k\right)=\omega_m'\left(k\right)+i\omega_m''\left(k\right)$$

,

$$\mathbf{\Psi}(t,z)=\sum_{m=1}^{n}\mathbf{\Psi}_m(t,z)=\sum_{m=1}^{n}A_m\vec{\Phi}_m(k)\exp\left[\omega_m''\left(k\right)t\right]\exp\left[-i\omega_m'(k)t+ikz\right]$$

$$V_{\Phi}^{(m)}=\frac{\operatorname{Re}\omega_m(k)}{k}=\frac{\omega_m'(k)}{k}$$

$$\boldsymbol{\Psi}(t,z)=\sum_{m=1}^{n}\boldsymbol{\Psi}_m(t,z)=\sum_{m=1}^{n}A_m\begin{pmatrix}\phi_1^{(m)}(k)\\ \phi_2^{(m)}(k)\\ \vdots\\ \phi_n^{(m)}(k)\end{pmatrix}\exp\left[-i\omega_m(k)t+ikz\right].$$

$$\boldsymbol{\Psi}(0,z)=\begin{pmatrix}b_1(\chi)\\ b_2(\chi)\\ \vdots\\ b_n(\chi)\end{pmatrix}\exp(i\chi z)$$

$$\sum_{m=1}^{n}A_m\begin{pmatrix}\phi_1^{(m)}(k)\\ \phi_2^{(m)}(k)\\ \vdots\\ \phi_n^{(m)}(k)\end{pmatrix}=\begin{pmatrix}b_1(k)\\ b_2(k)\\ \vdots\\ b_n(k)\end{pmatrix}\longrightarrow\sum_{m=1}^{n}A_m(k)\begin{pmatrix}1\\ L_2\left(\omega_m(k),k\right)\\ \vdots\\ L_n\left(\omega_m(k),k\right)\end{pmatrix}=\begin{pmatrix}b_1(k)\\ b_2(k)\\ \vdots\\ b_n(k)\end{pmatrix}$$

,

$$A_1+A_2=E_0$$

$$A_1-A_2=\frac{1}{\sqrt{\varepsilon_0}}B_0$$

$$\boldsymbol{\Psi}(t,z)=\begin{Bmatrix}E_x\\ B_y\end{Bmatrix}=\frac{1}{2}\left(E_0+\frac{1}{\sqrt{\varepsilon_0}}B_0\right)\begin{pmatrix}1\\ \sqrt{\varepsilon_0}\end{pmatrix}\exp\left[ik(z-c_0t)\right]+\frac{1}{2}\left(E_0-\frac{1}{\sqrt{\varepsilon_0}}B_0\right)\begin{pmatrix}1\\ -\sqrt{\varepsilon_0}\end{pmatrix}\exp\left[ik(z+c_0t)\right]$$

$$\frac{\partial}{\partial t} \to -i\omega, \qquad \frac{\partial}{\partial z} \to ik,$$
$$\int (\ )dt \to i\omega^{-1}, \quad \int (\ )dz \to -ik^{-1}.$$

$$\frac{\partial \psi_s}{\partial t} + \sum_{j=1}^{n}\left(A_{sj}\frac{\partial \psi_j}{\partial z} + B_{sj}\psi_j \right) = f_s(t,z), \qquad s = 1,2,\ldots,n$$

,

$$\mathbf{\Psi}(t,z) = \begin{pmatrix} 1 \\ L_2(\omega,k) \\ \vdots \\ L_n(\omega,k) \end{pmatrix} A(\omega,k)\exp(-i\omega t + ikz)$$

.

$$D(\omega,k)A(\omega,k)\exp(-i\omega t + ikz) = 0$$

$$\vec{\Psi}(t,z) = \{E_x, B_y\} = \left\{1, \frac{\varepsilon_0}{c}\frac{\omega}{k}\right\} A(\omega,k)\exp(-i\omega t + ikz)$$

$$\left(-\omega^2 + k^2 c_0^2\right) A(\omega,k)\exp(-i\omega t + ikz) = 0$$

$$A(\omega,k)\exp(-i\omega t + ikz) \to A(t,z)$$

$$\hat{\omega} = i\frac{\partial}{\partial t}, \quad \hat{k} = -i\frac{\partial}{\partial z} \qquad \hat{\omega}^{-1} = -i\int (\ )dt, \quad \hat{k}^{-1} = i\int (\ )dz$$

$$\mathbf{\Psi}(t,z) = \begin{pmatrix} 1 \\ L_2(\hat{\omega},\hat{k}) \\ \vdots \\ L_n(\hat{\omega},\hat{k}) \end{pmatrix} A(t,z)$$

$$D(\hat{\omega},\hat{k})A(t,z) = 0 \qquad D(\hat{\omega},\hat{k})A(t,z) = F(t,z)$$

$$\vec{\Psi}(t,z) = \{E_x(t,z), B_y(t,z)\} = \left\{ A(t,z), \ -\frac{\varepsilon_0}{c}\int \frac{\partial A}{\partial t}(t,z)dz \right\}$$

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \frac{\partial^2}{\partial z^2} \right) A(t,z) = 0$$

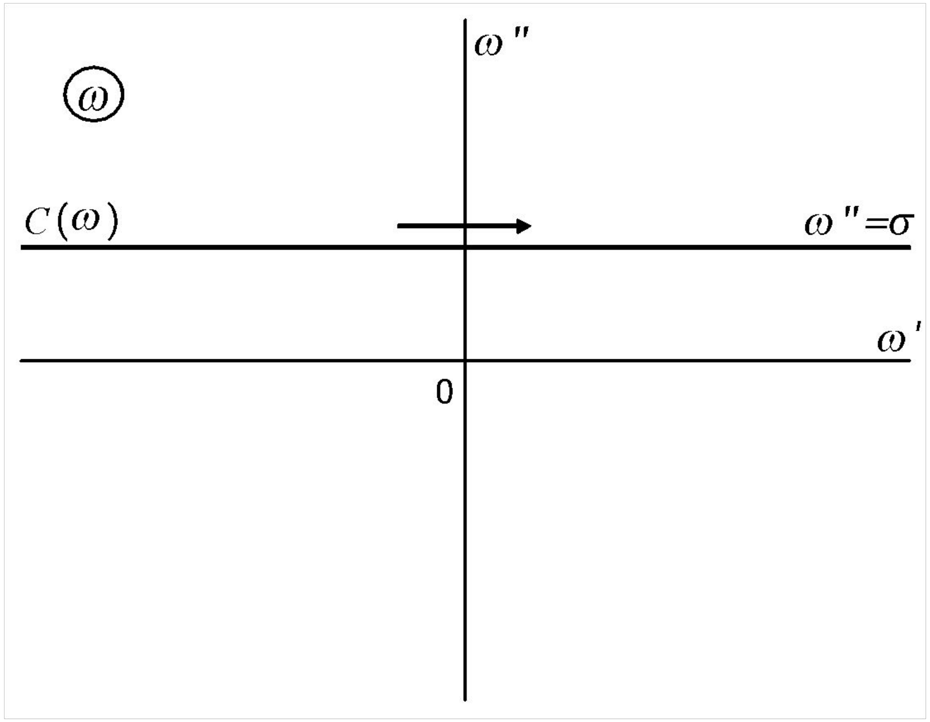
$$\varphi(\omega) = \int_0^\infty \varphi(t)\exp(i\omega\, t)dt$$

$$\varphi'(\omega) = -i\omega\varphi(\omega) - \varphi(t=0) \qquad \varphi^{(n)}(\omega) = (-i\omega)^n\left[\varphi(\omega) - \sum_{q=1}^{n}\frac{\varphi^{(q-1)}(t=0)}{(-i\omega)^q}\right]$$

, $$\phi(t) = \int_0^t \varphi(\tau)d\tau \;\Rightarrow \phi(\omega) = \frac{i}{\omega}\varphi(\omega)$$

$$S(t) = \int_0^t \varphi_1(\tau)\varphi_2(t-\tau)d\tau = \int_0^t \varphi_1(t-\tau)\varphi_2(\tau)d\tau \Rightarrow \; S(\omega) = \varphi_1(\omega)\varphi_2(\omega)$$

$$\varphi(t) = \frac{1}{2\pi}\int_{C(\omega)} \varphi(\omega)\exp(-i\omega\, t)d\omega$$



$$D(\hat{\omega}, k)A(t,k) = F(t,k)$$

$$A(t,k) = \frac{1}{2\pi} \int_{C(\omega)} \frac{P_{n-1}(\omega,k)}{D(\omega,k)} \exp(-i\omega\, t)d\omega$$

, $$A(t,k) = \sum_{m=1}^{n} \left[A_m(k)\exp\left(-i\omega_m(k)t\right)\right]$$

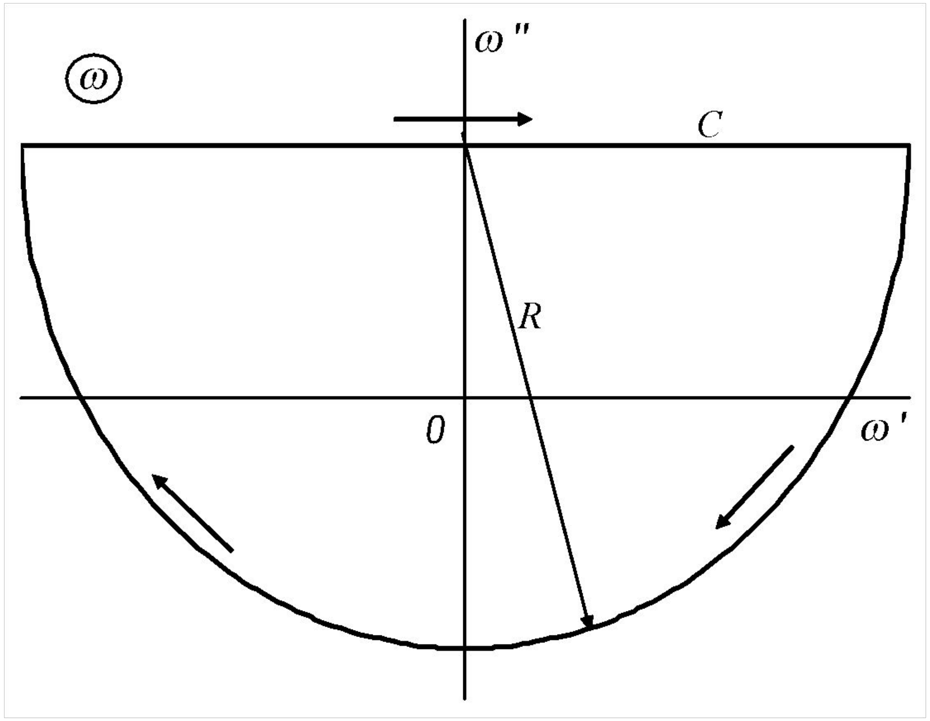
$$D(\omega,k)A(\omega,k) = F(\omega,k)$$

$$\downarrow$$

$$A(\omega,k) = G(\omega,k)F(\omega,k)$$

$$A(t,k) = \int_0^t G(\tau,k)F(t-\tau,k)d\tau$$

$$G(t,k) = \frac{1}{2\pi} \int_{C(\omega)} \frac{1}{D(\omega,k)} \exp(-i\omega\, t)d\omega$$



$$\frac{\partial E_x}{\partial t} + \frac{c}{\varepsilon_0}\frac{\partial B_y}{\partial z} = 0,$$

$$\frac{\partial B_y}{\partial t} + c\frac{\partial E_x}{\partial z} = 0,$$

$$\{E_x(t,z), B_y(t,z)\} = \{\, E_x(t,k),\; B_y(t,k)\}\exp(ikz)$$

$$\frac{dE_x}{dt} + i\frac{c}{\varepsilon_0}kB_y = 0,$$

$$\frac{dB_y}{dt} + ickE_x = 0.$$

$$-i\omega\, E_x + i(c/\varepsilon_0)k\, B_y = E_0,$$

$$ick\, E_x - i\omega\, B_y = B_0.$$

$$E_x(\omega,k) = -i\frac{\omega E_0 + kc/\varepsilon_0\, B_0}{D(\omega,k)},$$

$$B_y(\omega,k) = -i\frac{\omega B_0 + kcE_0}{D(\omega,k)}$$

$$E_x(t,z) = \frac{i}{2\pi}\int_{C(\omega)} \exp(-i\omega\, t)\frac{\omega E_0 + kc/\varepsilon_0\, B_0}{(\omega - kc_0)(\omega + kc_0)}d\omega\,,$$

$$B_y(t,z) = \frac{i}{2\pi}\int_{C(\omega)} \exp(-i\omega\, t)\frac{\omega B_0 + kcE_0}{(\omega - kc_0)(\omega + kc_0)}d\omega\,,$$