CEE 451G ENVIRONMENTAL FLUID MECHANICS

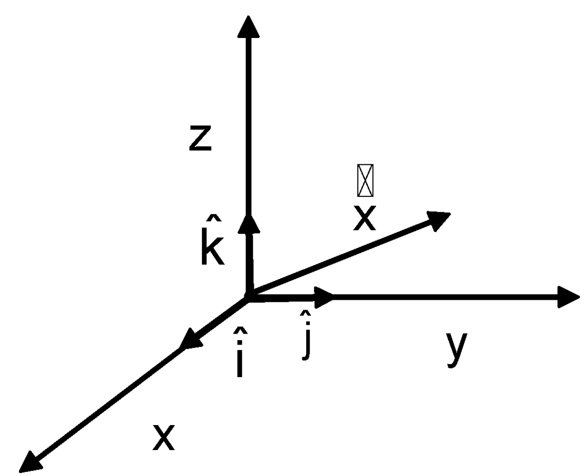
LECTURE 1: SCALARS, VECTORS AND TENSORS

A **scalar** has magnitude but no direction.
An example is pressure p.
The coordinates x, y and z of Cartesian space are scalars.

A **vector** has both magnitude and direction
Let $\hat{i}, \hat{j}, \hat{k}$ denote **unit** vectors in the x, y and z direction. The **hat** denotes a magnitude of unity

The **position vector** $\vec{x}$ (the arrow denotes a vector that is not a unit vector) is given as

$$\vec{x} = x\hat{i} + y\hat{j} + z\hat{k}$$



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The **velocity vector** $\vec{u}$ is given as

$$\vec{u} = \frac{d\vec{x}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

The **acceleration** vector $\vec{a}$ is given as

$$\vec{a} = \frac{d\vec{u}}{dt} = \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k} = \frac{d^2\vec{x}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

The **units** that we will use in class are length L, time T, mass M and temperature °. The units of a parameter are denoted in brackets. Thus

$$[\vec{x}] = L$$

$$[\vec{u}] = LT^{-1}$$

$$[\vec{a}] = ? \qquad LT^{-2}$$

Newton's second law is a vectorial statement: where $\vec{F}$ denotes the force vector and m denotes the mass (which is a scalar)

$$\vec{F} = m\vec{a}$$

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The components of the force vector can be written as follows:

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

The **dimensions** of the force vector are the dimension of mass times the dimension acceleration

$$[\vec{F}] = [F_x] = MLT^{-2}$$

Pressure p, which is a scalar, has dimensions of force per unit area. The dimensions of pressure are thus

$$[p] = MLT^{-2} / (L^2) = ML^{-1}T^{-2}$$

The acceleration of gravity g is a scalar with the dimensions of (of course) acceleration:

$$[g] = LT^{-2}$$

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A scalar can be a function of a vector, a vector of a scalar, etc. For example, in fluid flows pressure and velocity are both functions of position and time:

$$p = p(\vec{x}, t) \quad , \quad \vec{u} = \vec{u}(\vec{x}, t)$$

A scalar is a **zero-order tensor**. A vector is a **first-order tensor**. A matrix is a **second order tensor**. For example, consider the **stress tensor τ**.

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

The stress tensor has 9 components. What do they mean? Use the following mnemonic device: **first face, second stress**

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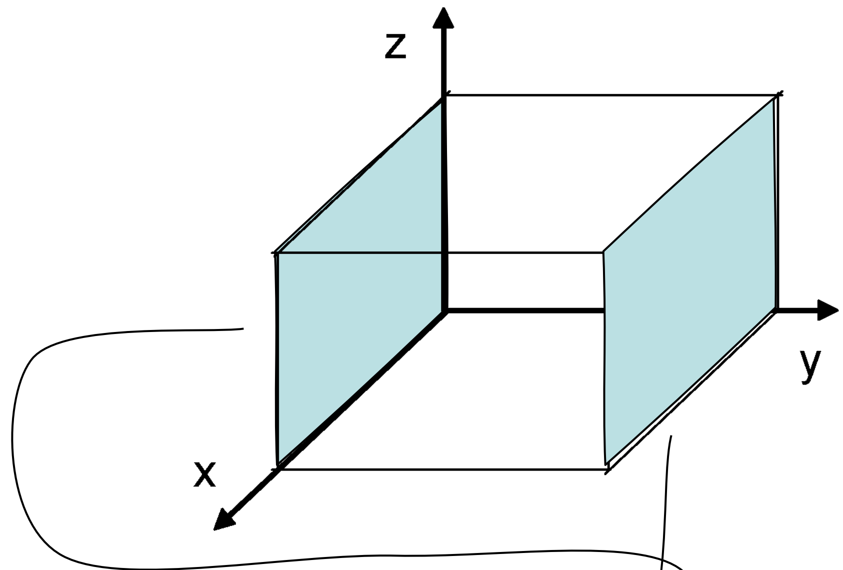
Consider the volume element below.



Each of the six faces has a **direction**.
For example, this face
and this face
are normal to the y direction

A force acting on any face can act in the x, y and z directions.

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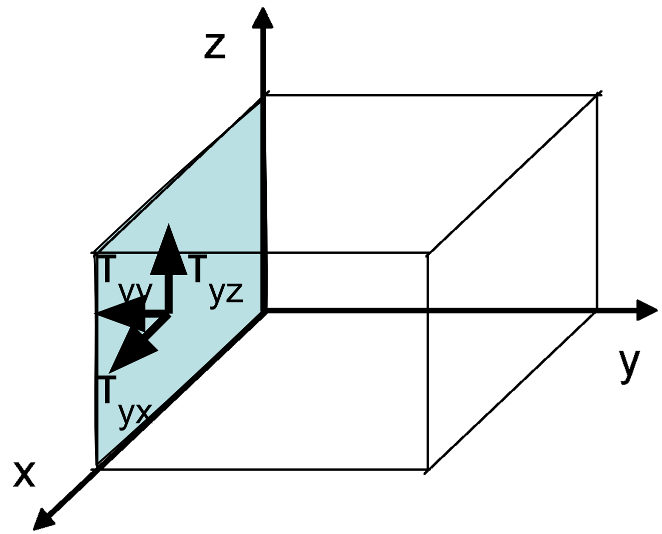
Consider the face below.



The face is in the direction y.

The force per unit face area acting in the x direction on that face is the stress τ_{yx} (first face, second stress).

The forces per unit face area acting in the y and z directions on that face are the stresses τ_{yy} and τ_{yz}.

Here τ_{yy} is a **normal stress** (acts normal, or perpendicular to the face) and τ_{yx} and τ_{yz} are **shear stresses** (act parallel to the face)

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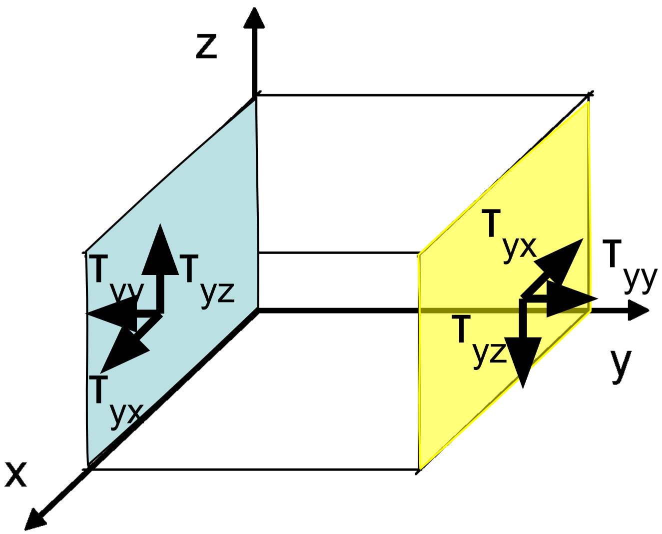
Some conventions are in order



Normal stresses are defined to be positive **outward**, so the orientation is reversed on the face located Δy from the origin

Shear stresses similarly reverse sign on the opposite face face are the stresses τ_{yy} and τ_{yz}.

Thus a positive normal stress puts a body in tension, and a negative normal stress puts the body in compression. Shear stresses always put the body in shear.`

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Another way to write a vector is in **Cartesian** form:

$$\vec{x} = x\hat{i} + y\hat{j} + z\hat{k} = (x, y, z)$$

The coordinates x, y and z can also be written as x_1, x_2, x_3. Thus the vector can be written as

$$\vec{x} = (x_1, x_2, x_3)$$

or as

$$\vec{x} = (x_i)\ , i = 1..3$$

or in **index notation**, simply as

$$\vec{x} = x_i$$

where i is understood to be a dummy variable running from 1 to 3.

Thus x_i, x_j and x_p all refer to the same vector (x_1, x_2 and x_3) , as the index (subscript) always runs from 1 to 3.

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Scalar multiplication: let α be a scalar and $\vec{A} = A_i$ be a vector. Then

$$\alpha\vec{A} = \alpha A_i = (\alpha A_i, \alpha A_2, \alpha A_3)$$

is a vector.

Dot or scalar product of two vectors results in a scalar:

$$\vec{A} \bullet \vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = \text{scalar}$$

In index notation, the dot product takes the form

$$\vec{A} \bullet \vec{B} = \sum_{i=1}^{3} A_iB_i = \sum_{k=1}^{3} A_kB_k = \sum_{r=1}^{3} A_rB_r =$$

Einstein summation convention: if the same index occurs twice, **always sum over that index**. So we abbreviate to

$$\vec{A} \bullet \vec{B} = A_iB_i = A_kB_k = A_rB_r$$

There is no free index in the above expressions. Instead the indices are paired (e.g. two i's), implying summation. The result of the dot product is thus a scalar.

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Magnitude of a vector:

$$\left|\vec{A}\right|^2 = \vec{A} \bullet \vec{A} = A_i A_i$$

A **tensor** can be constructed by multiplying two vectors (not scalar product):

$$A_i B_j = (A_i B_j)\ ,i = 1..3,\ j = 1..3 = \begin{pmatrix} A_1B_1 & A_2B_1 & A_3B_1 \\ A_1B_2 & A_2B_2 & A_3B_3 \\ A_1B_3 & A_2B_3 & A_3B_3 \end{pmatrix}$$

Two free indices (i, j) means the result is a **second-order** tensor

Now consider the expression

$$A_i A_j B_j$$

This is a **first-order tensor**, or **vector** because there is only one free index, i (the j's are paired, implying summation).

$$A_i A_j B_j = (A_1B_1 + A_2B_2 + A_3B_2)(A_1, A_2, A_3)$$

That is, scalar times vector = vector.

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Kronecker delta δ_{ij}

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since there are two free indices, the result is a second-order tensor, or matrix. The Kronecker delta corresponds to the **identity matrix**.

Third-order **Levi-Civita** tensor.

$$\varepsilon_{ijl} = \begin{cases} 1 & \text{if} \quad i,j,k & \text{cycle clockwise: 1,2,3, 2,3,1 or 3,1,2} \\ -1 & \text{if} \quad i,j,k & \text{cycle counterclockwise: 1,3,2, 3,2,2 or 2,1,3} \\ 0 & & \text{otherwise} \end{cases}$$

Vectorial **cross product**:

$$\vec{A} x \vec{B} = \varepsilon_{ijk} A_j B_k$$

One free index, so the result must be a **vector**.

Vectorial cross product: Let $\vec{C}$ be given as

$$\vec{C} = \vec{A} x \vec{B}$$

Then

$$\vec{C} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} =$$

$$\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}\begin{pmatrix} \hat{i} & \hat{j} \\ A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} - \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}\begin{pmatrix} \hat{i} & \hat{j} \\ A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} =$$

$$(A_2B_3 - A_3B_2)\hat{i} + (A_3B_1 - A_1B_3)\hat{j} + (A_1B_2 - A_2B_1)\hat{k}$$

Vectorial cross product in tensor notation:

$$C_i = \varepsilon_{ijk} A_j B_k$$

Thus for example

$$C_1 = \varepsilon_{1jk} A_j B_k = \underbrace{\varepsilon_{123}}_{=1} A_2 B_3 + \underbrace{\varepsilon_{132}}_{=-1} A_3 B_2 + \underbrace{\varepsilon_{111}}_{=0} A_1 B_1 + \quad \text{a lot of other terms that all = 0}$$

$$= A_2 B_3 - A_3 B_2$$

i.e. the same result as the other slide. The same results are also obtained for C_2 and C_3.

The **nabla vector operator** $\vec{\nabla}$:

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3}$$

or in index notation

$$\frac{\partial}{\partial x_i}$$

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The **gradient** converts a scalar to a vector. For example, where p is pressure,

$$\mathrm{grad}(p) = \vec{\nabla} p = \frac{\partial p}{\partial x_1}\hat{i} + \frac{\partial p}{\partial x_2}\hat{j} + \frac{\partial p}{\partial x_3}\hat{k}$$

or in index notation

$$\mathrm{grad}(p) = \frac{\partial p}{\partial x_i}$$

The single free index i (free in that it is not paired with another i) in the above expression means that grad(p) is a vector.

The **divergence** converts a vector into a scalar. For example, where $\vec{u}$ is the velocity vector,

$$\mathrm{div}(\vec{u}) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_k}{\partial x_k}$$

Note that there is no free index (two i's or two k's), so the result is a scalar.

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The **curl** converts a vector to a vector. For example, where $\vec{u}$ is the velocity vector,

$$\mathrm{curl}(\vec{u}) = \nabla x \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} =$$

$$\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\hat{i} + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\hat{j} + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\hat{k}$$

or in index notation,

$$\mathrm{curl}(\vec{u}) = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

One free index i (the j's and the k's are paired) means that the result is a vector

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A useful manipulation in tensor notation can be used to change an index in an expression:

$$\delta_{ij} u_j = u_i$$

This manipulation works because the Kronecker delta $\delta_{ij} = 0$ except when $i = j$, in which case it equals 1.