

Формула Тейлора

Опр. $f(x)$ опред. в окр. x_0 , $n \in \mathbb{N}_0$. Многочлен

Тейлора $P_n(x, f)$ - многочлен степени $\leq n$, что

остаточной жеи $r_n(x, f) = f(x) - P_n(x, f)$:

$$r_n(x, f) = \bar{o}((x-x_0)^n) \text{ при } x \rightarrow x_0$$

Т1. $f(x): \exists f^{(n)}(x_0) \Rightarrow \exists u \in \mathcal{O}. P_n(x, f)$.

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + r_n(x, f)$$

Формула Тейлора с остаточным членом

$x_0 = 0$ - формула Маклорена.

$r_n(x, f) = \bar{o}((x-x_0)^n)$ ($x \rightarrow x_0$) - форма Пеано.

Для неё нужно $\exists f^{(n)}(x_0)$.

C_n. $f(x) : \exists f^{(n+1)}(x_0) \Rightarrow r_n(x, f) = \underline{\underline{O}}((x-x_0)^{n+1}) (x \rightarrow x_0)$

D-ва $f(x) = f(x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \underbrace{\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + r_{n+1}(x, f)}_{r_n(x, f)}$

$$\lim_{x \rightarrow x_0} \frac{r_n(x, f)}{(x-x_0)^{n+1}} = \lim_{x \rightarrow x_0} \left[\frac{f^{(n+1)}(x_0)}{(n+1)!} + \frac{r_{n+1}(x, f)}{(x-x_0)^{n+1}} \right] = \frac{f^{(n+1)}(x_0)}{(n+1)!}$$

$$\Rightarrow \left| \frac{r_n(x, f)}{(x-x_0)^{n+1}} \right| \leq C \text{ при } 0 < |x-x_0| < \eta, \text{ т. е.}$$

$$r_n(x, f) = \underline{\underline{O}}((x-x_0)^{n+1}) (x \rightarrow x_0)$$

Сл. доказано.

$r_n(x, f) = \underline{\underline{O}}((x-x_0)^{n+1}) (x \rightarrow x_0)$ - улучшенная форма Пеано

Для неё нужно $\exists f^{(n+1)}(x_0)$

Т.2. $f(x): f^{(n)}(x)$ не пр. в окр. x_0 , $\exists f^{(n+1)}(x)$ в окр.

окр. x_0 , $x \in \dot{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0 \Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

Д-во Т.2. $\varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \dots - \frac{f^{(n)}(t)}{n!}(x-t)^n$

$$\varphi(t) \Big|_{t=x_0} = r_n(x, f), \quad \varphi(t) \Big|_{t=x} = 0, \quad \varphi(t) \in C[x_0, x], \quad \varphi'(t) =$$

$$= -f'(t) + f'(t) - \frac{f''(t)}{1!}(x-t) + \dots + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n \Rightarrow (\text{Т. Коши}) \exists \xi \in (x_0, x):$$

$$\frac{\varphi(x) - \varphi(x_0)}{\varphi'(x) - \varphi'(x_0)} = \frac{\psi'(x) - \psi'(x_0)}{\psi'(x) - \psi'(x_0)} \Rightarrow r_n(x, f) = \varphi(t) \Big|_{t=x_0} = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} (x - \xi)^n$$

Т. Доразана

Т.2. $f(x): f^{(n)}(x)$ непрерыв. в окр. x_0 , $\exists f^{(n+1)}(x)$ в прел.

окр. x_0 , $x \in \dot{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0 \Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x-t)^p, \quad p \in \mathbb{N} \quad \psi'(t) = -p(x-t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шлешильгх-Росс}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1} \quad \text{— глобальная оценка}$$

Д-во Т.2. $\varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \dots - \frac{f^{(n)}(t)}{n!}(x-t)^n$

$$\varphi(t)|_{t=x_0} = r_n(x, f), \quad \varphi(t)|_{t=x} = 0, \quad \varphi(t) \in C[x_0, x], \quad \varphi'(t) =$$

$$= -f'(t) + f'(t) - \frac{f''(t)}{1!}(x-t) + \dots + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n =$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n \Rightarrow (\text{т. Коши}) \exists \xi \in (x_0, x):$$

$$\frac{\varphi(x) - \varphi(x_0)}{\psi(x) - \psi(x_0)} = \frac{\varphi'(\xi)}{\psi'(\xi)} \Rightarrow r_n(x, f) = \varphi(t)|_{t=x_0} = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} (x-\xi)^n$$

Т. доказана

$$\Theta = \frac{\xi - x_0}{x - x_0} \in (0, 1), \quad x - \xi = (x - x_0)(1 - \Theta) \Rightarrow$$

$$r_n(x, f) = \frac{f^{(n+1)}(x_0 + \Theta(x-x_0))}{n!} (1-\Theta)^n (x-x_0)^{n+1} \quad (\text{Коши})$$

Т.2. $f(x): f^{(n)}(x)$ непрерывна в окр. x_0 , $\exists f^{(n+1)}(x)$ в прок.

окр. x_0 , $x \in \dot{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0 \Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x-t)^p, p \in \mathbb{N} \quad \psi'(t) = -p(x-t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шлемильх-Рочу}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1} \quad \text{глобальная оценка}$$

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + r_n(x, f) \quad \underline{x_0=0}$$

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + r_n(x, f) = \sum_{k=0}^n \frac{x^k}{k!} + r_n(x, f)$$

Д-во Т.2. $\varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!} (x-t) - \dots - \frac{f^{(n)}(t)}{n!} (x-t)^n$

$$\varphi(t)|_{t=x_0} = r_n(x, f), \quad \varphi(t)|_{t=x} = 0, \quad \varphi(t) \in C[x_0, x], \quad \varphi'(t) =$$

$$= -f'(t) + f'(t) - \frac{f''(t)}{1!} (x-t) + \dots + \frac{f^{(n)}(t)}{(n-1)!} (x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!} (x-t)^n = -\frac{f^{(n+1)}(t)}{n!} (x-t)^n \Rightarrow (\text{Т. Коши}) \exists \xi \in (x_0, x):$$

$$\frac{\varphi(x) - \varphi(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{\varphi'(t)}{\varphi'(t)} \Rightarrow r_n(x, f) = \varphi(t)|_{t=x_0} = \frac{\varphi(x) - \varphi(x_0)}{\varphi'(t)} \cdot \frac{f^{(n+1)}(t)}{n!} (x-t)^n$$

Т. доказано

$$\theta = \frac{\xi - x_0}{x - x_0} \in (0, 1), \quad x - \xi = (x - x_0)(1 - \theta) \Rightarrow$$

$$r_n(x, f) = \frac{f^{(n+1)}(x_0 + \theta(x-x_0))}{n!} (1-\theta)^n (x-x_0)^{n+1} \quad (\text{Коши})$$

$$1. f(x) = e^x, f^{(n)}(x) = e^x, f^{(n)}(0) = 1$$

$$r_n(x, f) = \bar{O}(x^n) \quad (x \rightarrow 0) \quad |r_n(x, f)| \leq \frac{e^A \cdot A^{n+1}}{(n+1)!}, \quad -A \leq x \leq A, \quad A > 0$$

Т.2. $f(x): f^{(n)}(x)$ непрерыв. в окр. x_0 , $\exists f^{(n+1)}(x)$ в прок. окр. x_0 , $x \in \dot{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$ $\exists \psi'(t) \neq 0$. $\Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x-t)^p, p \in \mathbb{N} \quad \psi'(t) = -p(x-t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шле́йлинг - Роль}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1} \quad \text{глобальная оценка}$$

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + r_n(x, f) \quad x_0 = 0$$

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + r_n(x, f) = \sum_{k=0}^n \frac{x^k}{k!} + r_n(x, f)$$

$$2. \sin x = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x, f)$$

Д-во Т.2. $\varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!} (x-t) - \dots - \frac{f^{(n)}(t)}{n!} (x-t)^n$

$$\varphi(t) \Big|_{t=x_0} = r_n(x, f), \quad \varphi(t) \Big|_{t=x} = 0, \quad \varphi(t) \in C[x_0, x], \quad \varphi'(t) =$$

$$= -f'(t) + f'(t) - \frac{f''(t)}{1!} (x-t) + \dots + \frac{f^{(n)}(t)}{(n-1)!} (x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!} (x-t)^n =$$

$$= -\frac{f^{(n+1)}(t)}{n!} (x-t)^n \Rightarrow (\text{Т. Коши}) \exists \xi \in (x_0, x):$$

$$\frac{\varphi(x) - \varphi(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{\varphi'(\xi)}{\psi'(\xi)} \Rightarrow r_n(x, f) = \varphi(t) \Big|_{t=x_0} = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} (x-\xi)^n$$

Т. доказано

$$\Theta = \frac{\xi - x_0}{x - x_0} \in (0, 1), \quad x - \xi = (x - x_0)(1 - \Theta) \Rightarrow$$

$$r_n(x, f) = \frac{f^{(n+1)}(x_0 + \Theta(x-x_0))}{n!} (1-\Theta)^n (x-x_0)^{n+1} \quad (\text{Коши})$$

$$1. f(x) = e^x, f^{(n)}(x) = e^x, f^{(n)}(0) = 1$$

$$r_n(x, f) = o(x^n) \quad (x \rightarrow 0) \quad |r_n(x, f)| \leq \frac{e^A \cdot A^{n+1}}{(n+1)!}, \quad -A \leq x \leq A, \quad A > 0$$

$$2. f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f^{(3)}(x) = \sin x, \dots$$

$$f^{(m)}(0) = 0, \quad m = 0, 2, 4, 6, \dots; \quad r_{2n+1}(x, f) = r_{2n+2}(x, f) = o(x^{2n+2})$$

$$|r_{2n+1}(x, f)| = |r_{2n+2}(x, f)| \leq \frac{A^{2n+3}}{(2n+3)!} \quad |x| \leq A$$

Т. 2. $f(x): f^{(n)}(x)$ непрерыв. в окр. $x_0, \exists f^{(n+1)}(x)$ в прел.

окр. $x_0, x \in \dot{U}(x_0), \psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0. \Rightarrow \exists \xi \in (x_0, x):$

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x-t)^p, p \in \mathbb{N} \quad \psi'(t) = -p(x-t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шлёмлюх-Росс}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1} \quad \text{— глобальная оценка}$$

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + r_n(x, f) \quad x_0 = 0$$

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + r_n(x, f) = \sum_{k=0}^n \frac{x^k}{k!} + r_n(x, f)$$

$$2. \sin x = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x, f)$$

$$3. \cos x = 1 - \frac{x^2}{2!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n}(x, f)$$

2-во Т. 2. $\varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!} (x-t) - \dots - \frac{f^{(n)}(t)}{n!} (x-t)^n$

$$\varphi(t) \Big|_{t=x_0} = r_n(x, f), \quad \varphi(t) \Big|_{t=x} = 0, \quad \varphi(t) \in C[x_0, x], \quad \varphi'(t) =$$

$$= -f'(t) + f'(t) - \frac{f''(t)}{1!} (x-t) + \dots + \frac{f^{(n)}(t)}{(n-1)!} (x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!} (x-t)^n =$$

$$= -\frac{f^{(n+1)}(t)}{n!} (x-t)^n \Rightarrow (\text{Т. Коши}) \exists \xi \in (x_0, x):$$

$$\frac{\varphi(x) - \varphi(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{\varphi'(t)}{\varphi'(t)} \Rightarrow r_n(x, f) = \varphi(t) \Big|_{t=x_0} = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} (x-\xi)^n$$

Т. доказано

$$\theta = \frac{\xi - x_0}{x - x_0} \in (0, 1), \quad x - \xi = (x - x_0)(1 - \theta) \Rightarrow$$

$$r_n(x, f) = \frac{f^{(n+1)}(x_0 + \theta(x-x_0))}{n!} (1-\theta)^n (x-x_0)^{n+1} \quad (\text{Коши})$$

1. $f(x) = e^x, f^{(n)}(x) = e^x, f^{(n)}(0) = 1$

$$r_n(x, f) = \bar{o}(x^n) \quad (x \rightarrow 0) \quad |r_n(x, f)| \leq \frac{e^A \cdot A^{n+1}}{(n+1)!}, \quad -A \leq x \leq A, \quad A > 0$$

2. $f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f^{(3)}(x) = \sin x, \dots$

$$f^{(m)}(0) = 0, \quad m = 0, 2, 4, 6, \dots; \quad |r_{2n+1}(x, f)| = |r_{2n+2}(x, f)| = \bar{o}(x^{2n+2})$$

$$|r_{2n+1}(x, f)| = |r_{2n+2}(x, f)| \leq \frac{A^{2n+3}}{(2n+3)!} \quad |x| \leq A$$

3. Сделать самостоятельно и получить: $|r_{2n}(x, f)| = |r_{2n+1}(x, f)| = \bar{o}(x^{2n+1})$

$$|r_{2n}(x, f)| = |r_{2n+1}(x, f)| \leq \frac{A^{2n+2}}{(2n+2)!} \quad |x| \leq A$$

Т.2. $f(x): f^{(n)}(x)$ непрерывна в окр. x_0 , $\exists f^{(n+1)}(x)$ в прел.

окр. x_0 , $x \in \bar{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0 \Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x-t)^p, p \in \mathbb{N} \quad \psi'(t) = -p(x-t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шлемиль-Росс}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1} \quad \text{глобальная оценка}$$

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + r_n(x, f) \quad x_0 = 0$$

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + r_n(x, f) = \sum_{k=0}^n \frac{x^k}{k!} + r_n(x, f)$$

$$2. \sin x = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x, f)$$

$$3. \cos x = 1 - \frac{x^2}{2!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n}(x, f)$$

$$4. (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2} x^2 + \dots + \frac{\alpha(\alpha-1) \dots (\alpha-n+1)}{1 \cdot 2 \cdot \dots \cdot n} x^n + r_n(x, f)$$

$$\text{Д-во Т.2. } \varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!} (x-t) - \dots - \frac{f^{(n)}(t)}{n!} (x-t)^n$$

$$\varphi(t) \Big|_{t=x_0} = r_n(x, f), \quad \varphi(t) \Big|_{t=x} = 0, \quad \varphi(t) \in C[x_0, x], \quad \varphi'(t) =$$

$$= -f'(t) + f'(t) - \frac{f''(t)}{1!} (x-t) + \dots + \frac{f^{(n)}(t)}{(n-1)!} (x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!} (x-t)^n = -\frac{f^{(n+1)}(t)}{n!} (x-t)^n \Rightarrow (\text{Т. Коши}) \exists \xi \in (x_0, x);$$

$$\frac{\varphi(x) - \varphi(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{\varphi'(\xi)}{\psi'(\xi)} \Rightarrow r_n(x, f) = \varphi(t) \Big|_{t=x_0} = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} (x-\xi)^n$$

Т. доказано

$$\Theta = \frac{\xi - x_0}{x - x_0} \in (0, 1), \quad x - \xi = (x - x_0)(1 - \Theta) \Rightarrow$$

$$r_n(x, f) = \frac{f^{(n+1)}(x_0 + \Theta(x-x_0))}{n!} (1-\Theta)^n (x-x_0)^{n+1} \quad (\text{Коши})$$

1. $f(x) = e^x, f^{(n)}(x) = e^x, f^{(n)}(0) = 1$

$$r_n(x, f) = \bar{0}(x^n) \quad (x \rightarrow 0) \quad |r_n(x, f)| \leq \frac{e^A \cdot A^{n+1}}{(n+1)!}, \quad -A \leq x \leq A, \quad A > 0$$

2. $f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = \sin x, \dots$

$$f^{(m)}(0) = 0, \quad m = 0, 2, 4, 6, \dots; \quad r_{2n+1}(x, f) = r_{2n+2}(x, f) = \bar{0}(x^{2n+2})$$

$$|r_{2n+1}(x, f)| = |r_{2n+2}(x, f)| \leq \frac{A^{2n+3}}{(2n+3)!} \quad |x| \leq A$$

3. Сделать самообразование и получить: $r_{2n}(x, f) = r_{2n+1}(x, f) = \bar{0}(x^{2n+1})$

$$|r_{2n}(x, f)| = |r_{2n+1}(x, f)| \leq \frac{A^{2n+2}}{(2n+2)!} \quad |x| \leq A$$

4. $f(x) = (1+x)^\alpha, f'(x) = \alpha(1+x)^{\alpha-1}, \dots, f^{(n)}(x) = \alpha(\alpha-1) \dots (\alpha-n+1)(1+x)^{\alpha-n}$

$$r_n(x, f) = \bar{0}(x^n). \quad \alpha \in \mathbb{N}_0 \Rightarrow \text{Символ Ньютона}$$

Т.2. $f(x): f^{(m)}(x)$ непрерыв. в окр. x_0 , $\exists f^{(n+1)}(x)$ в прок.

окр. x_0 , $x \in \dot{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0 \Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x-t)^p, p \in \mathbb{N} \quad \psi'(t) = -p(x-t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шлемильх-Росс}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1} \quad \text{глобальная оценка}$$

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + r_n(x, f) \quad x_0 = 0$$

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + r_n(x, f) = \sum_{k=0}^n \frac{x^k}{k!} + r_n(x, f)$$

$$2. \sin x = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x, f) = \sum_{k=0}^n \frac{(-1)^k x^{2k+1}}{(2k+1)!} + r_{2n+1}(x, f)$$

$$3. \cos x = 1 - \frac{x^2}{2!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x, f) = \sum_{k=0}^n \frac{(-1)^k x^{2k}}{(2k)!} + r_{2n}(x, f)$$

$$4. (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2} x^2 + \dots + \frac{\alpha(\alpha-1) \dots (\alpha-n+1)}{1 \cdot 2 \cdot \dots \cdot n} x^n + r_n(x, f)$$

$$5. \ln(1+x) = x - \frac{x^2}{2} + \dots + (-1)^{n-1} \frac{x^n}{n} + r_n(x, f) = \sum_{k=1}^n \frac{(-1)^{k-1} x^k}{k} + r_n(x, f)$$

$$\text{Д-во Т.2. } \varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!} (x-t) - \dots - \frac{f^{(n)}(t)}{n!} (x-t)^n$$

$$\varphi(t) \Big|_{t=x_0} = r_n(x, f), \quad \varphi(t) \Big|_{t=x} = 0, \quad \varphi(t) \in C[x_0, x], \quad \varphi'(t) =$$

$$= -f'(t) + f'(t) - \frac{f''(t)}{1!} (x-t) + \dots + \frac{f^{(n)}(t)}{(n-1)!} (x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!} (x-t)^n =$$

$$= -\frac{f^{(n+1)}(t)}{n!} (x-t)^n \Rightarrow (\text{Т. Коши}) \exists \xi \in (x_0, x);$$

$$\frac{\varphi(x) - \varphi(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{\varphi'(\xi)}{\varphi'(\xi)} \Rightarrow r_n(x, f) = \varphi(t) \Big|_{t=x_0} = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} (x-\xi)^n$$

Т. доказана

$$\theta = \frac{\xi - x_0}{x - x_0} \in (0, 1), \quad x - \xi = (x - x_0)(1 - \theta) \Rightarrow$$

$$r_n(x, f) = \frac{f^{(n+1)}(x_0 + \theta(x-x_0))}{n!} (1-\theta)^n (x-x_0)^{n+1} \quad (\text{Коши})$$

$$1. f(x) = e^x, f'(x) = e^x, f^{(n)}(0) = 1$$

$$r_n(x, f) = \bar{o}(x^n) \quad (x \rightarrow 0) \quad |r_n(x, f)| \leq \frac{e^A \cdot A^{n+1}}{(n+1)!}, \quad -A \leq x \leq A, \quad A > 0$$

$$2. f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f^{(3)}(x) = \sin x, \dots$$

$$f^{(m)}(0) = 0, \quad m = 0, 2, 4, 6, \dots; \quad r_{2n+1}(x, f) = r_{2n+2}(x, f) = \bar{o}(x^{2n+2})$$

$$|r_{2n+1}(x, f)| = |r_{2n+2}(x, f)| \leq \frac{A^{2n+3}}{(2n+3)!} \quad |x| \leq A$$

$$3. \text{Сделать самостоятельно и проверить: } r_{2n}(x, f) = r_{2n+1}(x, f) = \bar{o}(x^{2n+1})$$

$$|r_{2n}(x, f)| = |r_{2n+1}(x, f)| \leq \frac{A^{2n+2}}{(2n+2)!} \quad |x| \leq A$$

$$4. f(x) = (1+x)^\alpha, f'(x) = \alpha(1+x)^{\alpha-1}, \dots, f^{(n)}(x) = \alpha(\alpha-1) \dots (\alpha-n+1)(1+x)^{\alpha-n}$$

$$r_n(x, f) = \bar{o}(x^n). \quad \alpha \in \mathbb{N}_0 \Rightarrow \text{Случай Ньютона}$$

$$5. f(x) = \ln(1+x), f'(x) = \frac{1}{1+x}, f''(x) = -\frac{1}{(1+x)^2}, \dots, f^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{(1+x)^n}$$

$$r_n(x, f) = \bar{o}(x^n)$$

Т.2. $f(x): f^{(n)}(x)$ непрерывна в окр. x_0 , $\exists f^{(n+1)}(x)$ в прок.

окр. x_0 , $x \in \dot{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0. \Rightarrow \exists \xi \in (x_0, x):$

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x - t)^p, p \in \mathbb{N} \quad \psi'(t) = -p(x - t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x - x_0)^p (x - \xi)^{n+1-p} \quad \text{Шлешильгх-Рочу}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x - x_0)(x - \xi)^n \quad \text{Коши} \quad \leftarrow$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x - x_0|^{n+1} \quad \text{глобальная оценка}$$

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + r_n(x, f) \quad x_0 = 0$$

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + r_n(x, f) = \sum_{k=0}^n \frac{x^k}{k!} + r_n(x, f)$$

$$2. \sin x = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x, f)$$

$$3. \cos x = 1 - \frac{x^2}{2!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n}(x, f)$$

$$4. (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2} x^2 + \dots + \frac{\alpha(\alpha-1) \dots (\alpha-n+1)}{1 \cdot 2 \cdot \dots \cdot n} x^n + r_n(x, f)$$

$$5. \ln(1+x) = x - \frac{x^2}{2} + \dots + (-1)^{n-1} \frac{x^n}{n} + r_n(x, f) = \sum_{k=1}^n (-1)^{k-1} \frac{x^k}{k} + r_n(x, f)$$

Примеры

$$1. \lim_{x \rightarrow 0} \frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = 1$$

$$\frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = \frac{\frac{x^3}{6} + o(x^4)}{\frac{x^3}{6} + o(x^3)} = \frac{\frac{1}{6} + x \cdot \frac{o(x^4)}{x^4}}{\frac{1}{6} + \frac{o(x^3)}{x^3}} \rightarrow 1$$

T.2. $f(x): f^{(n)}(x)$ непрерывна в окр. x_0 , $\exists f^{(n+1)}(x)$ в прок.

окр. x_0 , $x \in \dot{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0 \Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x-t)^p, p \in \mathbb{N} \quad \psi'(t) = -p(x-t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шлешильх-Рочу}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1} \quad \text{глобальная оценка}$$

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + r_n(x, f) \quad x_0 = 0$$

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + r_n(x, f) = \sum_{k=0}^n \frac{x^k}{k!} + r_n(x, f)$$

$$2. \sin x = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x, f)$$

$$3. \cos x = 1 - \frac{x^2}{2!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n}(x, f)$$

$$4. (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2} x^2 + \dots + \frac{\alpha(\alpha-1) \dots (\alpha-n+1)}{1 \cdot 2 \dots n} x^n + r_n(x, f)$$

$$5. \ln(1+x) = x - \frac{x^2}{2} + \dots + (-1)^{n-1} \frac{x^n}{n} + r_n(x, f) = \sum_{k=1}^n (-1)^{k-1} \frac{x^k}{k} + r_n(x, f)$$

Примеры

$$1. \lim_{x \rightarrow 0} \frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = 1$$

$$\frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = \frac{x^3/6 + o(x^4)}{x^3/6 + o(x^3)} = \frac{1/6 + x \cdot \frac{o(x^4)}{x^4}}{1/6 + \frac{o(x^3)}{x^3}} \rightarrow 1$$

$$2. \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{2}} - e}{x} = -\frac{e}{2}$$

$$\frac{(1+x)^{\frac{1}{2}} - e}{x} = \frac{e^{\frac{1}{2} \ln(1+x)} - e}{x} = \frac{e^{\frac{1}{2} [x - \frac{x^2}{2} + o(x^2)]} - e}{x} =$$

$$= \frac{\left(\frac{1}{x} \frac{o(x^2)}{x} \text{ есть } o(x), \right.}{\left. \text{т.к. } \frac{(1/2) \cdot o(x^2)}{x} = \frac{o(x^4)}{x^2} \rightarrow 0 \right)} = \frac{e^{1 - \frac{x}{2} + o(x)} - e}{x} = \frac{e \left[e^{-\frac{x}{2} + o(x)} - 1 \right]}{x}$$

$$= \frac{e \cdot \left[1 - \frac{x}{2} + o(x) + o\left(-\frac{x}{2} + o(x)\right) - 1 \right]}{x} = \frac{e \cdot \left(o\left(-\frac{x}{2} + o(x)\right) = o\left(-\frac{x}{2}\right) = \bar{o}(x) \right)}{x}$$

$$= \frac{e \cdot \left[-\frac{x}{2} + o(x) \right]}{x} = e \left[-\frac{1}{2} + \frac{o(x)}{x} \right] \rightarrow -\frac{e}{2}$$

1.2. $f(x): f^{(n)}(x)$ непрерывна в окр. x_0 , $\exists f^{(n+1)}(x)$ в прех.

окр. x_0 , $x \in \dot{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0 \Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x-t)^p, p \in \mathbb{N} \quad \psi'(t) = -p(x-t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шлешильс-Рохс}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1} \quad \text{глобальная оценка}$$

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + r_n(x, f) \quad x_0 = 0$$

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + r_n(x, f) = \sum_{k=0}^n \frac{x^k}{k!} + r_n(x, f)$$

$$2. \sin x = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x, f)$$

$$3. \cos x = 1 - \frac{x^2}{2!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n}(x, f)$$

$$4. (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2} x^2 + \dots + \frac{\alpha(\alpha-1) \dots (\alpha-n+1)}{1 \cdot 2 \cdot \dots \cdot n} x^n + r_n(x, f)$$

$$5. \ln(1+x) = x - \frac{x^2}{2} + \dots + (-1)^{n-1} \frac{x^n}{n} + r_n(x, f) = \sum_{k=1}^n (-1)^{k-1} \frac{x^k}{k} + r_n(x, f)$$

$\dots \dots \dots f'(t) \dots \dots \dots f^{(n)}(t) \dots \dots$

Примеры

$$1. \lim_{x \rightarrow 0} \frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = 1$$

$$\frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = \frac{x^3/6 + o(x^4)}{x^3/6 + o(x^3)} = \frac{1/6 + x \cdot \frac{o(x^4)}{x^4}}{1/6 + \frac{o(x^3)}{x^3}} \rightarrow 1$$

$$2. \lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - e}{x} = -\frac{e}{2}$$

$$\frac{(1+x)^{1/x} - e}{x} = \frac{e^{\frac{\ln(1+x)}{x}} - e}{x} = \frac{e^{\frac{1}{x}[x - \frac{x^2}{2} + o(x^2)]} - e}{x} =$$

$$= \frac{\left(\frac{1}{x} o(x^2) \text{ есть } o(x), \right.}{\left. \text{т.к. } \left(\frac{1}{x} o(x^2) - \frac{o(x)}{x^2} \rightarrow 0 \right) \right)}{x} = \frac{e^{1 - \frac{x}{2} + o(x)} - e}{x} = \frac{e \left[e^{-\frac{x}{2} + o(x)} - 1 \right]}{x}$$

$$= \frac{e \cdot \left[1 - \frac{x}{2} + o(x) + o\left(-\frac{x}{2} + o(x)\right) - 1 \right]}{x} = \left(o\left(-\frac{x}{2} + o(x)\right) = o\left(-\frac{x}{2}\right) = o(x) \right)$$

$$= \frac{e \cdot \left[-\frac{x}{2} + o(x) \right]}{x} = e \left[-\frac{1}{2} + \frac{o(x)}{x} \right] \rightarrow -\frac{e}{2}$$

$$3. \lim_{n \rightarrow \infty} \left[\frac{\left(1 + \frac{1}{n}\right)^n}{e} \right]^n =$$

Т.2. $f(x): f^{(n)}(x)$ непр. в окр. x_0 , $\exists f^{(n+1)}(x)$ в прок.

окр. x_0 , $x \in \dot{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0. \Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x-t)^p, \quad p \in \mathbb{N} \quad \psi'(t) = -p(x-t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шлеилера - Роуи}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1} \quad \text{- глобальная оценка}$$

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + r_n(x, f) \quad x_0 = 0$$

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + r_n(x, f) = \sum_{k=0}^n \frac{x^k}{k!} + r_n(x, f)$$

$$2. \sin x = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x, f)$$

$$3. \cos x = 1 - \frac{x^2}{2!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n}(x, f)$$

$$4. (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2} x^2 + \dots + \frac{\alpha(\alpha-1) \dots (\alpha-n+1)}{1 \cdot 2 \cdot \dots \cdot n} x^n + r_n(x, f)$$

$$5. \ln(1+x) = x - \frac{x^2}{2} + \dots + (-1)^{n-1} \frac{x^n}{n} + r_n(x, f) = \sum_{k=1}^n (-1)^{k-1} \frac{x^k}{k} + r_n(x, f)$$

Примеры

$$1. \lim_{x \rightarrow 0} \frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = 1$$

$$\frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = \frac{\frac{x^3}{6} + o(x^4)}{\frac{x^3}{6} + o(x^3)} = \frac{\frac{1}{6} + x \cdot \frac{o(x^4)}{x^4}}{\frac{1}{6} + \frac{o(x^3)}{x^3}} \rightarrow 1$$

$$2. \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{e}{2}$$

$$\frac{(1+x)^{\frac{1}{x}} - e}{x} = \frac{e^{\frac{\ln(1+x)}{x}} - e}{x} = \frac{e^{\frac{1}{x} [x - \frac{x^2}{2} + o(x^2)]} - e}{x} =$$

$$= \frac{\left(\frac{\frac{1}{x} o(x^2)}{x} e^{o(x)} + o(x), \right)}{\frac{1}{x} \cdot \left(\frac{1}{x} \cdot \frac{o(x^2)}{x} = \frac{o(x^2)}{x^2} \rightarrow 0 \right)} = \frac{e^{1 - \frac{x}{2} + o(x)} - e}{x} = \frac{e \left[e^{-\frac{x}{2} + o(x)} - 1 \right]}{x}$$

$$= \frac{e \cdot \left[1 - \frac{x}{2} + o(x) + o\left(-\frac{x}{2} + o(x)\right) - 1 \right]}{x} = \frac{o\left(-\frac{x}{2} + o(x)\right) = o\left(-\frac{x}{2}\right) = -\frac{x}{2}}{x}$$

$$= \frac{e \cdot \left[-\frac{x}{2} + o(x) \right]}{x} = e \left[-\frac{1}{2} + \frac{o(x)}{x} \right] \rightarrow -\frac{e}{2}$$

$$3. \lim_{n \rightarrow \infty} \left[\frac{\left(1 + \frac{1}{n}\right)^n}{e} \right]^n = \frac{1}{\sqrt{e}}$$

$$\left(\frac{1 + \frac{1}{n}}{e} \right)^{n^2} = e^{n^2 \ln\left(1 + \frac{1}{n}\right) - n} = e^{n^2 \left[\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \right] - n} =$$

$$= e^{-\frac{1}{2} + \frac{o\left(\frac{1}{n^2}\right)}{1/n^2}} \rightarrow e^{-\frac{1}{2}}$$

Т.2. $f(x): f^{(n)}(x)$ непр. в окр. x_0 , $\exists f^{(n+1)}(x)$ в прок.
окр. x_0 , $x \in \dot{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$

$\exists \psi'(t) \neq 0 \Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

$$\psi(t) = (x-t)^p, p \in \mathbb{N} \quad \psi'(t) = -p(x-t)^{p-1}$$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!} (x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шлешильсх-Росс}$$

$$p = n+1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!} (x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формулы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!} |x-x_0|^{n+1} \quad \text{- глобальная оценка}$$

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + r_n(x, f) \quad x_0 = 0$$

$$1. e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + r_n(x, f) = \sum_{k=0}^n \frac{x^k}{k!} + r_n(x, f)$$

$$2. \sin x = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x, f)$$

$$3. \cos x = 1 - \frac{x^2}{2!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x, f) = \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n}(x, f)$$

$$4. (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2} x^2 + \dots + \frac{\alpha(\alpha-1) \dots (\alpha-n+1)}{1 \cdot 2 \cdot \dots \cdot n} x^n + r_n(x, f)$$

$$5. \ln(1+x) = x - \frac{x^2}{2} + \dots + (-1)^{n-1} \frac{x^n}{n} + r_n(x, f) = \sum_{k=1}^n (-1)^{k-1} \frac{x^k}{k} + r_n(x, f)$$

4. $f(x) = \operatorname{ctg} x - \frac{1}{x}$ Найми главный член вида Cx^α при $x \rightarrow 0$.

$$\operatorname{ctg} x - \frac{1}{x} = \frac{\cos x}{\sin x} - \frac{1}{x} = \frac{x \cos x - \sin x}{x \cdot \sin x} =$$

$$= \frac{x \left(1 - \frac{x^2}{2} + \bar{o}(x^3)\right) - \left(x - \frac{x^3}{6} + \bar{o}(x^4)\right)}{x \cdot (x + \bar{o}(x^2))} =$$

$$= \frac{x - \frac{x^3}{2} + \bar{o}(x^4) - x + \frac{x^3}{6} + \bar{o}(x^4)}{x^2 + \bar{o}(x^3)} = \frac{-\frac{x^3}{3} + \bar{o}(x^4)}{x^2 + \bar{o}(x^3)}$$

$$\sim -\frac{x}{3}, \text{ т.к. } \frac{-\frac{x^3}{3} + \bar{o}(x^4)}{x^2 + \bar{o}(x^3)} \div \left(-\frac{x}{3}\right) =$$

$$= \frac{x^2 + \left(-\frac{3}{x}\right) \cdot \bar{o}(x^4)}{x^2 + \bar{o}(x^3)} = \frac{1 + (-3) \cdot x \cdot \frac{\bar{o}(x^4)}{x^4}}{1 + x \cdot \frac{\bar{o}(x^3)}{x^3}} \rightarrow 1 (x \rightarrow 0)$$