Формула Тейлора

Опр. $f(x)$ опред. в окр. x_0, $n \in \mathbb{N}_0$. Многочлен Тейлора $P_n(x, f)$ – многочлен степени $\leq n$, что остаточный член $r_n(x,f) = f(x) - P_n(x,f)$:

$$r_n(x,f) = \overline{\overline{o}}\left((x-x_0)^n\right) \text{ при } x \to x_0$$

T1. $f(x)$: $\exists f^{(n)}(x_0) \Rightarrow \exists$ и ед. $P_n(x,f)$.

$$\boxed{f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + r_n(x,f)}$$

Формула Тейлора с остаточным членом

$x_0 = 0$ – формула Маклорена.

$r_n(x,f) = \overline{\overline{o}}\left((x-x_0)^n\right)$ $(x \to x_0)$ – форма Пеано.

Для неё нужно $\exists f^{(n)}(x_0)$.

Сл. $f(x): \exists f^{(n+1)}(x_0) \Rightarrow r_n(x,f) = \underline{\underline{O}}((x-x_0)^{n+1})\ (x\to x_0)$

Д-во $f(x) = f(x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \underbrace{\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + \overbrace{r_{n+1}(x,f)}^{\overline{\overline{o}}((x-x_0)^{n+1})}}_{r_n(x,f)}$,

$$\lim_{x\to x_0}\frac{r_n(x,f)}{(x-x_0)^{n+1}} = \lim_{x\to x_0}\left[\frac{f^{(n+1)}(x_0)}{(n+1)!} + \frac{r_{n+1}(x,f)}{(x-x_0)^{n+1}}\right] = \frac{f^{(n+1)}(x_0)}{(n+1)!}$$

$\Rightarrow \left|\frac{r_n(x,f)}{(x-x_0)^{n+1}}\right| \leqslant C$ при $0<|x-x_0|<\eta$, т.е.

$r_n(x,f) = \underline{\underline{O}}((x-x_0)^{n+1})\ (x\to x_0)$

Сл. доказано.

$r_n(x,f) = \underline{\underline{O}}((x-x_0)^{n+1})\ (x\to x_0)$ – улучшенная форма Пеано

Для неё нужно $\exists f^{(n+1)}(x_0)$

Т. 2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x \in \mathring{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$ $\exists \psi'(t) \neq 0$. $\Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x - \xi)^n$$

Д-во т. 2 $\varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \ldots - \frac{f^{(n)}(t)}{n!}(x-t)^n$

$\varphi(t)\big|_{t=x_0} = r_n(x, f)$, $\varphi(t)\big|_{t=x} = 0$, $\varphi(t) \in C[x_0, x]$, $\varphi'(t) =$

$= -f'(t) + f'(t) - \frac{f''(t)}{1!}(x-t) + \ldots + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n =$

$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$. $\Rightarrow$ (т. Коши) $\exists \xi \in (x_0, x)$:

$$\frac{\varphi(x) - \varphi(x_0)}{\psi(x) - \psi(x_0)} = \frac{\varphi'(\xi)}{\psi'(\xi)} \Rightarrow r_n(x, f) = \varphi(t)\big|_{t=x_0} = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$$

Т. доказана

Т. 2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x \in \mathring{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$ $\exists \psi'(t) \neq 0$. $\Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x, f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x-\xi)^n$$

$\psi(t) = (x-t)^p$, $p \in \mathbb{N}$ $\quad \psi'(t) = -p(x-t)^{p-1}$

$$r_n(x, f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!}(x-x_0)^p (x-\xi)^{n+1-p}$$ Шлёмильх – Рош

$p = n+1$ $\quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ Лагранж

$p = 1$ $\quad r_n(x, f) = \frac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n$ Коши

Из формы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$\Rightarrow |r_n(x, f)| \leq \frac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ – глобальная оценка

Д-во т. 2. $\varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \ldots - \frac{f^{(n)}(t)}{n!}(x-t)^n$

$\varphi(t)\big|_{t=x_0} = r_n(x, f)$, $\varphi(t)\big|_{t=x} = 0$, $\varphi(t) \in C[x_0, x]$, $\varphi'(t) =$

$= -f'(t) + f'(t) - \frac{f''(t)}{1!}(x-t) + \ldots + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n =$

$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$. $\Rightarrow$ (т. Коши) $\exists \xi \in (x_0, x)$:

$$\frac{\varphi(x) - \varphi(x_0)}{\psi(x) - \psi(x_0)} = \frac{\varphi'(\xi)}{\psi'(\xi)} \Rightarrow r_n(x, f) = \varphi(t)\big|_{t=x_0} = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$$

Т. доказана

$\Theta = \frac{\xi - x_0}{x - x_0} \in (0, 1)$, $x - \xi = (x-x_0)(1-\Theta) \Rightarrow$

$$r_n(x, f) = \frac{f^{(n+1)}(x_0 + \Theta(x-x_0))}{n!}(1-\Theta)^n (x-x_0)^{n+1} \quad \text{(Коши)}$$

Т.2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x \in \mathring{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$ $\exists \psi'(t) \neq 0$. $\Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x,f) = \frac{\psi(x)-\psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x-\xi)^n$$

$\psi(t) = (x-t)^p$, $p \in \mathbb{N}$ $\quad \psi'(t) = -p(x-t)^{p-1}$

$$r_n(x,f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!}(x-x_0)^p (x-\xi)^{n+1-p}$$ Шлёмильх - Рош

$p = n+1$ $\quad r_n(x,f) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ Лагранж

$p = 1$ $\quad r_n(x,f) = \frac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n$ Коши

Из формы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$\Rightarrow |r_n(x,f)| \leq \frac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ – глобальная оценка

<u>Основные формулы</u>

$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + r_n(x,f)$ $\quad$ <u>$x_0 = 0$</u>

1. $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + r_n(x,f) = \sum\limits_{k=0}^{n} \frac{x^k}{k!} + r_n(x,f)$

Д-во т.2. $\varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \ldots - \frac{f^{(n)}(t)}{n!}(x-t)^n$

$\varphi(t)\big|_{t=x_0} = r_n(x,f)$, $\varphi(t)\big|_{t=x} = 0$, $\varphi(t) \in C[x_0, x]$, $\varphi'(t) =$

$= -f'(t) + f'(t) - \frac{f''(t)}{1!}(x-t) + \ldots + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n =$

$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$. $\Rightarrow$ (т. Коши) $\exists \xi \in (x_0, x)$:

$$\frac{\varphi(x)-\varphi(x_0)}{\psi(x)-\psi(x_0)} = \frac{\varphi'(\xi)}{\psi'(\xi)} \Rightarrow r_n(x,f) = \varphi(t)\big|_{t=x_0} = \frac{\psi(x)-\psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$$

Т. доказана

$\Theta = \frac{\xi - x_0}{x - x_0} \in (0,1)$, $x - \xi = (x-x_0)(1-\Theta) \Rightarrow$

$$r_n(x,f) = \frac{f^{(n+1)}(x_0 + \Theta(x-x_0))}{n!}(1-\Theta)^n (x-x_0)^{n+1}$$ (Коши)

1. $f(x) = e^x$, $f^{(n)}(x) = e^x$, $f^{(n)}(0) = 1$

$r_n(x,f) = \bar{\bar{o}}(x^n)$ $(x \to 0)$ $\quad |r_n(x,f)| \leq \frac{e^A \cdot A^{n+1}}{(n+1)!}$, $-A \leq x \leq A$, $A > 0$

Т. 2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x\in \mathring{U}(x_0)$, $\psi(t)\in C[x_0,x]$ и $\forall t\in(x_0,x)$ $\exists\, \psi'(t)\neq 0$. $\Rightarrow \exists\, \xi\in(x_0,x)$:

$$r_n(x,f)=\frac{\psi(x)-\psi(x_0)}{\psi'(\xi)}\cdot\frac{f^{(n+1)}(\xi)}{n!}\cdot(x-\xi)^n$$

$\psi(t)=(x-t)^p$, $p\in\mathbb{N}$ $\quad\psi'(t)=-p(x-t)^{p-1}$

$r_n(x,f)=\dfrac{f^{(n+1)}(\xi)}{p\cdot n!}(x-x_0)^p(x-\xi)^{n+1-p}$ Шлёмильх – Рош

$p=n+1$ $\quad r_n(x,f)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ Лагранж

$p=1$ $\quad r_n(x,f)=\dfrac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n$ Коши

Из формы Лагранжа: если $|f^{(n+1)}(x)|\leq M_{n+1}\Rightarrow$

$\Rightarrow |r_n(x,f)|\leq \dfrac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ – глобальная оценка

Основные формулы

$f(x)=f(x_0)+\dfrac{f'(x_0)}{1!}(x-x_0)+\ldots+\dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x,f)$ $\quad x_0=0$

1. $e^x=1+x+\dfrac{x^2}{2!}+\ldots+\dfrac{x^n}{n!}+r_n(x,f)=\sum\limits_{k=0}^{n}\dfrac{x^k}{k!}+r_n(x,f)$

2. $\sin x=x-\dfrac{x^3}{3!}+\ldots+(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}+r_{2n+1}(x,f)=\sum\limits_{k=0}^{n}(-1)^k\dfrac{x^{2k+1}}{(2k+1)!}+r_{2n+1}(x,f)$

Д-во т. 2. $\varphi(t)=f(x)-f(t)-\dfrac{f'(t)}{1!}(x-t)-\ldots-\dfrac{f^{(n)}(t)}{n!}(x-t)^n$

$\varphi(t)\big|_{t=x_0}=r_n(x,f)$, $\varphi(t)\big|_{t=x}=0$, $\varphi(t)\in C[x_0,x]$, $\varphi'(t)=$

$=-f'(t)+f'(t)-\dfrac{f''(t)}{1!}(x-t)+\ldots+\dfrac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}-\dfrac{f^{(n+1)}(t)}{n!}(x-t)^n=$

$=-\dfrac{f^{(n+1)}(t)}{n!}(x-t)^n$. $\Rightarrow$ (т. Коши) $\exists\,\xi\in(x_0,x)$:

$\dfrac{\varphi(x)-\varphi(x_0)}{\psi(x)-\psi(x_0)}=\dfrac{\varphi'(\xi)}{\psi'(\xi)}\Rightarrow r_n(x,f)=\varphi(t)\big|_{t=x_0}=\dfrac{\psi(x)-\psi(x_0)}{\psi'(\xi)}\cdot\dfrac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$

Т. доказана

$\Theta=\dfrac{\xi-x_0}{x-x_0}\in(0,1)$, $x-\xi=(x-x_0)(1-\Theta)\Rightarrow$

$r_n(x,f)=\dfrac{f^{(n+1)}(x_0+\Theta(x-x_0))}{n!}(1-\Theta)^n(x-x_0)^{n+1}$ (Коши)

1. $f(x)=e^x$, $f^{(n)}(x)=e^x$, $f^{(n)}(0)=1$

$r_n(x,f)=\bar{\bar{o}}(x^n)$ $(x\to 0)$ $\quad |r_n(x,f)|\leq\dfrac{e^A\cdot A^{n+1}}{(n+1)!}$, $-A\leq x\leq A$, $A>0$

2. $f(x)=\sin x$, $f'(x)=\cos x$, $f''(x)=-\sin x$, $f^{IV}(x)=\sin x$, ...

$f^{(m)}(0)=0$, $m=0,2,4,6,\ldots$; $r_{2n+1}(x,f)=r_{2n+2}(x,f)=\bar{\bar{o}}(x^{2n+2})$

$|r_{2n+1}(x,f)|=|r_{2n+2}(x,f)|\leq\dfrac{A^{2n+3}}{(2n+3)!}$ $\quad |x|\leq A$

Т.2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x\in \mathring{U}(x_0)$, $\psi(t)\in C[x_0,x]$ и $\forall t\in(x_0,x)$ $\exists\, \psi'(t)\neq 0$. $\Rightarrow \exists\, \xi\in(x_0,x)$:

$$r_n(x,f)=\frac{\psi(x)-\psi(x_0)}{\psi'(\xi)}\cdot\frac{f^{(n+1)}(\xi)}{n!}\cdot(x-\xi)^n$$

$\psi(t)=(x-t)^p$, $p\in\mathbb{N}$ $\quad\psi'(t)=-p(x-t)^{p-1}$

$r_n(x,f)=\dfrac{f^{(n+1)}(\xi)}{p\, n!}(x-x_0)^p(x-\xi)^{n+1-p}$ Шлёмильх-Рош

$p=n+1$ $\quad r_n(x,f)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ Лагранж

$p=1$ $\quad r_n(x,f)=\dfrac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n$ Коши

Из формы Лагранжа: если $|f^{(n+1)}(x)|\leq M_{n+1}\Rightarrow$

$\Rightarrow |r_n(x,f)|\leq \dfrac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ — глобальная оценка

Основные формулы

$f(x)=f(x_0)+\dfrac{f'(x_0)}{1!}(x-x_0)+\ldots+\dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x,f)$ $\quad x_0=0$

1. $e^x=1+x+\dfrac{x^2}{2!}+\ldots+\dfrac{x^n}{n!}+r_n(x,f)=\sum\limits_{k=0}^{n}\dfrac{x^k}{k!}+r_n(x,f)$

2. $\sin x=x-\dfrac{x^3}{3!}+\ldots+(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}+r_{2n+1}(x,f)=\sum\limits_{k=0}^{n}(-1)^k\dfrac{x^{2k+1}}{(2k+1)!}+r_{2n+1}(x,f)$

3. $\cos x=1-\dfrac{x^2}{2!}+\ldots+(-1)^n\dfrac{x^{2n}}{(2n)!}+r_{2n}(x,f)=\sum\limits_{k=0}^{n}(-1)^k\dfrac{x^{2k}}{(2k)!}+r_{2n}(x,f)$

Д-во т.2. $\varphi(t)=f(x)-f(t)-\dfrac{f'(t)}{1!}(x-t)-\ldots-\dfrac{f^{(n)}(t)}{n!}(x-t)^n$

$\varphi(t)\big|_{t=x_0}=r_n(x,f)$, $\varphi(t)\big|_{t=x}=0$, $\varphi(t)\in C[x_0,x]$, $\varphi'(t)=$

$=-f'(t)+f'(t)-\dfrac{f''(t)}{1!}(x-t)+\ldots+\dfrac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}-\dfrac{f^{(n+1)}(t)}{n!}(x-t)^n=$

$=-\dfrac{f^{(n+1)}(t)}{n!}(x-t)^n$. $\Rightarrow$ (т. Коши) $\exists\,\xi\in(x_0,x)$:

$\dfrac{\varphi(x)-\varphi(x_0)}{\psi(x)-\psi(x_0)}=\dfrac{\varphi'(\xi)}{\psi'(\xi)}\Rightarrow r_n(x,f)=\varphi(t)\big|_{t=x_0}=\dfrac{\psi(x)-\psi(x_0)}{\psi'(\xi)}\cdot\dfrac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$

Т. доказана

$\Theta=\dfrac{\xi-x_0}{x-x_0}\in(0,1)$, $x-\xi=(x-x_0)(1-\Theta)\Rightarrow$

$r_n(x,f)=\dfrac{f^{(n+1)}(x_0+\Theta(x-x_0))}{n!}(1-\Theta)^n(x-x_0)^{n+1}$ (Коши)

1. $f(x)=e^x$, $f^{(n)}(x)=e^x$, $f^{(n)}(0)=1$

$r_n(x,f)=\bar{\bar{o}}(x^n)$ $(x\to 0)$ $\quad |r_n(x,f)|\leq\dfrac{e^A\cdot A^{n+1}}{(n+1)!}$, $-A\leq x\leq A$, $A>0$

2. $f(x)=\sin x$, $f'(x)=\cos x$, $f''(x)=-\sin x$, $f^{IV}(x)=\sin x$, ...

$f^{(m)}(0)=0$, $m=0,2,4,6,\ldots$; $r_{2n+1}(x,f)=r_{2n+2}(x,f)=\bar{\bar{o}}(x^{2n+2})$

$|r_{2n+1}(x,f)|=|r_{2n+2}(x,f)|\leq\dfrac{A^{2n+3}}{(2n+3)!}$ $\quad |x|\leq A$

3. Сделать самостоятельно и получить: $r_{2n}(x,f)=r_{2n+1}(x,f)=\bar{\bar{o}}(x^{2n+1})$

$|r_{2n}(x,f)|=|r_{2n+1}(x,f)|\leq\dfrac{A^{2n+2}}{(2n+2)!}$ $\quad |x|\leq A$

Т. 2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x \in \mathring{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$ $\exists \psi'(t) \neq 0$. $\Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x,f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x-\xi)^n$$

$\psi(t) = (x-t)^p$, $p \in \mathbb{N}$ $\quad \psi'(t) = -p(x-t)^{p-1}$

$r_n(x,f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!}(x-x_0)^p (x-\xi)^{n+1-p}$ Шлёмильх-Рош

$p = n+1$ $\quad r_n(x,f) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ Лагранж

$p = 1$ $\quad r_n(x,f) = \frac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n$ Коши

Из формы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$\Rightarrow |r_n(x,f)| \leq \frac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ – глобальная оценка

Основные формулы

$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + r_n(x,f)$ $\quad x_0 = 0$

1. $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + r_n(x,f) = \sum_{k=0}^{n} \frac{x^k}{k!} + r_n(x,f)$
2. $\sin x = x - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x,f) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x,f)$
3. $\cos x = 1 - \frac{x^2}{2!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x,f) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n}(x,f)$
4. $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2}x^2 + \ldots + \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{1 \cdot 2 \cdot \ldots \cdot n}x^n + r_n(x,f)$

Д-во т. 2. $\quad \varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \ldots - \frac{f^{(n)}(t)}{n!}(x-t)^n$

$\varphi(t)\big|_{t=x_0} = r_n(x,f)$, $\varphi(t)\big|_{t=x} = 0$, $\varphi(t) \in C[x_0, x]$, $\varphi'(t) =$

$= -f'(t) + f'(t) - \frac{f''(t)}{1!}(x-t) + \ldots + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n =$

$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$. $\Rightarrow$ (т. Коши) $\exists \xi \in (x_0, x)$:

$\frac{\varphi(x) - \varphi(x_0)}{\psi(x) - \psi(x_0)} = \frac{\varphi'(\xi)}{\psi'(\xi)} \Rightarrow r_n(x,f) = \varphi(t)\big|_{t=x_0} = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$

Т. доказана

$\Theta = \frac{\xi - x_0}{x - x_0} \in (0,1)$, $\quad x - \xi = (x-x_0)(1-\Theta) \Rightarrow$

$r_n(x,f) = \frac{f^{(n+1)}(x_0 + \Theta(x-x_0))}{n!}(1-\Theta)^n (x-x_0)^{n+1}$ (Коши)

1. $f(x) = e^x$, $f^{(n)}(x) = e^x$, $f^{(n)}(0) = 1$

   $r_n(x,f) = \bar{\bar{o}}(x^n)$ $(x \to 0)$ $\quad |r_n(x,f)| \leq \frac{e^A \cdot A^{n+1}}{(n+1)!}$, $-A \leq x \leq A$, $A > 0$

2. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{IV}(x) = \sin x$, ...

   $f^{(m)}(0) = 0$, $m = 0, 2, 4, 6, \ldots$; $r_{2n+1}(x,f) = r_{2n+2}(x,f) = \bar{\bar{o}}(x^{2n+2})$

   $|r_{2n+1}(x,f)| = |r_{2n+2}(x,f)| \leq \frac{A^{2n+3}}{(2n+3)!}$ $\quad |x| \leq A$

3. Сделать самостоятельно и получить: $r_{2n}(x,f) = r_{2n+1}(x,f) = \bar{\bar{o}}(x^{2n+1})$

   $|r_{2n}(x,f)| = |r_{2n+1}(x,f)| \leq \frac{A^{2n+2}}{(2n+2)!}$ $\quad |x| \leq A$

4. $f(x) = (1+x)^\alpha$, $f'(x) = \alpha(1+x)^{\alpha-1}$, ..., $f^{(n)}(x) = \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)(1+x)^{\alpha-n}$

   $r_n(x,f) = \bar{\bar{o}}(x^n)$. $\quad \alpha \in \mathbb{N}_0 \Rightarrow$ Бином Ньютона

Т.2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x \in \overset{\circ}{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$ $\exists \psi'(t) \neq 0$. $\Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x,f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x-\xi)^n$$

$\psi(t) = (x-t)^p$, $p \in \mathbb{N}$ $\quad \psi'(t) = -p(x-t)^{p-1}$

$r_n(x,f) = \frac{f^{(n+1)}(\xi)}{p\, n!}(x-x_0)^p(x-\xi)^{n+1-p}$ Шлёмильх – Рош

$p = n+1$ $\quad r_n(x,f) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ Лагранж

$p = 1$ $\quad r_n(x,f) = \frac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n$ Коши

Из формы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$\Rightarrow |r_n(x,f)| \leq \frac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ – глобальная оценка

Основные формулы

$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + r_n(x,f)$ $\quad x_0 = 0$

1. $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + r_n(x,f) = \sum_{k=0}^{n}\frac{x^k}{k!} + r_n(x,f)$
2. $\sin x = x - \frac{x^3}{3!} + \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x,f) = \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x,f)$
3. $\cos x = 1 - \frac{x^2}{2!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + r_{2n}(x,f) = \sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!} + r_{2n}(x,f)$
4. $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1\cdot 2}x^2 + \ldots + \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{1\cdot 2\cdot\ldots\cdot n}x^n + r_n(x,f)$
5. $\ln(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + r_n(x,f) = \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k} + r_n(x,f)$

Д-во т.2. $\varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \ldots - \frac{f^{(n)}(t)}{n!}(x-t)^n$

$\varphi(t)\big|_{t=x_0} = r_n(x,f)$, $\varphi(t)\big|_{t=x} = 0$, $\varphi(t) \in C[x_0, x]$, $\varphi'(t) =$

$= -f'(t) + f'(t) - \frac{f''(t)}{1!}(x-t) + \ldots + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n =$

$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$. $\Rightarrow$ (т. Коши) $\exists \xi \in (x_0, x)$:

$\frac{\varphi(x)-\varphi(x_0)}{\psi(x)-\psi(x_0)} = \frac{\varphi'(\xi)}{\psi'(\xi)} \Rightarrow r_n(x,f) = \varphi(t)\big|_{t=x_0} = \frac{\psi(x)-\psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$

Т. доказана

$\Theta = \frac{\xi - x_0}{x - x_0} \in (0,1)$, $x - \xi = (x-x_0)(1-\Theta) \Rightarrow$

$r_n(x,f) = \frac{f^{(n+1)}(x_0 + \Theta(x-x_0))}{n!}(1-\Theta)^n(x-x_0)^{n+1}$ (Коши)

1. $f(x) = e^x$, $f^{(n)}(x) = e^x$, $f^{(n)}(0) = 1$

   $r_n(x,f) = \overline{\overline{o}}(x^n)$ $(x \to 0)$ $\quad |r_n(x,f)| \leq \frac{e^A \cdot A^{n+1}}{(n+1)!}$, $-A \leq x \leq A$, $A > 0$

2. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{IV}(x) = \sin x$, ...

   $f^{(m)}(0) = 0$, $m = 0, 2, 4, 6, \ldots$; $r_{2n+1}(x,f) = r_{2n+2}(x,f) = \overline{\overline{o}}(x^{2n+2})$

   $|r_{2n+1}(x,f)| = |r_{2n+2}(x,f)| \leq \frac{A^{2n+3}}{(2n+3)!}$ $\quad |x| \leq A$

3. Сделать самостоятельно и получить: $r_{2n}(x,f) = r_{2n+1}(x,f) = \overline{\overline{o}}(x^{2n+1})$

   $|r_{2n}(x,f)| = |r_{2n+1}(x,f)| \leq \frac{A^{2n+2}}{(2n+2)!}$ $\quad |x| \leq A$

4. $f(x) = (1+x)^\alpha$, $f'(x) = \alpha(1+x)^{\alpha-1}$, ..., $f^{(n)}(x) = \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)(1+x)^{\alpha-n}$

   $r_n(x,f) = \overline{\overline{o}}(x^n)$. $\alpha \in \mathbb{N}_0 \Rightarrow$ бином Ньютона

5. $f(x) = \ln(1+x)$, $f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$, ..., $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

   $r_n(x,f) = \overline{\overline{o}}(x^n)$

Т. 2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x \in \mathring{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$ $\exists \psi'(t) \neq 0$. $\Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x,f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x-\xi)^n$$

$\psi(t) = (x-t)^p$, $p \in \mathbb{N}$ $\quad \psi'(t) = -p(x-t)^{p-1}$

$r_n(x,f) = \dfrac{f^{(n+1)}(\xi)}{p \cdot n!}(x-x_0)^p(x-\xi)^{n+1-p}$ Шлёмильх-Роша

$p = n+1$ $\quad r_n(x,f) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ Лагранж

$p = 1$ $\quad r_n(x,f) = \dfrac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n$ Коши

Из формы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$\Rightarrow |r_n(x,f)| \leq \dfrac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ – глобальная оценка

Основные формулы

$f(x) = f(x_0) + \dfrac{f'(x_0)}{1!}(x-x_0) + \ldots + \dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n + r_n(x,f)$ $\quad \underline{x_0 = 0}$

1. $e^x = 1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^n}{n!} + r_n(x,f) = \sum\limits_{k=0}^{n} \dfrac{x^k}{k!} + r_n(x,f)$
2. $\sin x = x - \dfrac{x^3}{3!} + \ldots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x,f) = \sum\limits_{k=0}^{n} (-1)^k \dfrac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x,f)$
3. $\cos x = 1 - \dfrac{x^2}{2!} + \ldots + (-1)^n \dfrac{x^{2n}}{(2n)!} + r_{2n}(x,f) = \sum\limits_{k=0}^{n} (-1)^k \dfrac{x^{2k}}{(2k)!} + r_{2n}(x,f)$
4. $(1+x)^\alpha = 1 + \alpha x + \dfrac{\alpha(\alpha-1)}{1 \cdot 2}x^2 + \ldots + \dfrac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{1 \cdot 2 \cdot \ldots \cdot n}x^n + r_n(x,f)$
5. $\ln(1+x) = x - \dfrac{x^2}{2} + \ldots + (-1)^{n-1}\dfrac{x^n}{n} + r_n(x,f) = \sum\limits_{k=1}^{n} (-1)^{k-1}\dfrac{x^k}{k} + r_n(x,f)$

Примеры

1. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = 1$

$$\frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = \frac{x^3/6 + \bar{\bar{o}}(x^4)}{\frac{x^3}{6} + \bar{\bar{o}}(x^3)} = \frac{\frac{1}{6} + x \cdot \frac{\bar{\bar{o}}(x^4)}{x^4}}{\frac{1}{6} + \frac{\bar{\bar{o}}(x^3)}{x^3}} \to 1$$

Т.2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x\in \mathring{U}(x_0)$, $\psi(t)\in C[x_0,x]$ и $\forall t\in(x_0,x)$ $\exists \psi'(t)\neq 0$. $\Rightarrow \exists \xi\in(x_0,x)$:

$$r_n(x,f)=\frac{\psi(x)-\psi(x_0)}{\psi'(\xi)}\cdot\frac{f^{(n+1)}(\xi)}{n!}\cdot(x-\xi)^n$$

$\psi(t)=(x-t)^p$, $p\in\mathbb{N}$ $\quad \psi'(t)=-p(x-t)^{p-1}$

$r_n(x,f)=\dfrac{f^{(n+1)}(\xi)}{p\cdot n!}(x-x_0)^p(x-\xi)^{n+1-p}$ Шлёмильх-Рош

$p=n+1$ $\quad r_n(x,f)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ Лагранж

$p=1$ $\quad r_n(x,f)=\dfrac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n$ Коши

Из формы Лагранжа: если $|f^{(n+1)}(x)|\leq M_{n+1}\Rightarrow$

$\Rightarrow |r_n(x,f)|\leq \dfrac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ — глобальная оценка

Основные формулы

$f(x)=f(x_0)+\dfrac{f'(x_0)}{1!}(x-x_0)+\ldots+\dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x,f)$ $\quad x_0=0$

1. $e^x=1+x+\dfrac{x^2}{2!}+\ldots+\dfrac{x^n}{n!}+r_n(x,f)=\sum\limits_{k=0}^{n}\dfrac{x^k}{k!}+r_n(x,f)$
2. $\sin x=x-\dfrac{x^3}{3!}+\ldots+(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}+r_{2n+1}(x,f)=\sum\limits_{k=0}^{n}(-1)^k\dfrac{x^{2k+1}}{(2k+1)!}+r_{2n+1}(x,f)$
3. $\cos x=1-\dfrac{x^2}{2!}+\ldots+(-1)^n\dfrac{x^{2n}}{(2n)!}+r_{2n}(x,f)=\sum\limits_{k=0}^{n}(-1)^k\dfrac{x^{2k}}{(2k)!}+r_{2n}(x,f)$
4. $(1+x)^\alpha=1+\alpha x+\dfrac{\alpha(\alpha-1)}{1\cdot 2}x^2+\ldots+\dfrac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{1\cdot 2\cdot\ldots\cdot n}x^n+r_n(x,f)$
5. $\ln(1+x)=x-\dfrac{x^2}{2}+\ldots+(-1)^{n-1}\dfrac{x^n}{n}+r_n(x,f)=\sum\limits_{k=1}^{n}(-1)^{k-1}\dfrac{x^k}{k}+r_n(x,f)$

Примеры

1. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{e^x-1-x-\frac{x^2}{2}}=1$

$$\frac{x-\sin x}{e^x-1-x-\frac{x^2}{2}}=\frac{x^3/6+\bar{\bar{o}}(x^4)}{\frac{x^3}{6}+\bar{\bar{o}}(x^3)}=\frac{\frac{1}{6}+x\cdot\frac{\bar{\bar{o}}(x^4)}{x^4}}{\frac{1}{6}+\frac{\bar{\bar{o}}(x^3)}{x^3}}\to 1$$

2. $\lim\limits_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}-e}{x}=-\dfrac{e}{2}$

$$\frac{(1+x)^{\frac{1}{x}}-e}{x}=\frac{e^{\frac{\ln(1+x)}{x}}-e}{x}=\frac{e^{\frac{1}{x}\left[x-\frac{x^2}{2}+\bar{\bar{o}}(x^2)\right]}-e}{x}=$$

$$=\left(\begin{array}{l}\frac{1}{x}\bar{\bar{o}}(x^2)\text{ есть }\bar{\bar{o}}(x),\\ \text{т.к. }\frac{(1/x)\cdot\bar{\bar{o}}(x^2)}{x}=\frac{\bar{\bar{o}}(x^2)}{x^2}\to 0\end{array}\right)=\frac{e^{1-\frac{x}{2}+\bar{\bar{o}}(x)}-e}{x}=\frac{e\left[e^{-\frac{x}{2}+\bar{\bar{o}}(x)}-1\right]}{x}=$$

$$=\frac{e\cdot\left[1-\frac{x}{2}+\bar{\bar{o}}(x)+\bar{\bar{o}}\left(-\frac{x}{2}+\bar{\bar{o}}(x)\right)-1\right]}{x}=\left(\bar{\bar{o}}\left(-\frac{x}{2}+\bar{\bar{o}}(x)\right)=\bar{\bar{o}}\left(-\frac{x}{2}\right)=\bar{\bar{o}}(x)\right)$$

$$=\frac{e\cdot\left[-\frac{x}{2}+\bar{\bar{o}}(x)\right]}{x}=e\left[-\frac{1}{2}+\frac{\bar{\bar{o}}(x)}{x}\right]\to-\frac{e}{2}$$

$\underline{\text{Т. 2.}}$ $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x \in \overset{\circ}{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$ $\exists \psi'(t) \neq 0$. $\Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x,f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x-\xi)^n$$

$\psi(t) = (x-t)^p$, $p \in \mathbb{N}$ $\quad \psi'(t) = -p(x-t)^{p-1}$

$$r_n(x,f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!}(x-x_0)^p (x-\xi)^{n+1-p} \quad \text{Шлёмильх – Рош}$$

$$p = n+1 \quad r_n(x,f) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \quad \text{Лагранж}$$

$$p = 1 \quad r_n(x,f) = \frac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n \quad \text{Коши}$$

Из формы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$\Rightarrow |r_n(x,f)| \leq \frac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ – глобальная оценка

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + r_n(x,f) \quad \underline{x_0 = 0}$$

1. $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + r_n(x,f) = \sum_{k=0}^{n} \frac{x^k}{k!} + r_n(x,f)$
2. $\sin x = x - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x,f) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x,f)$
3. $\cos x = 1 - \frac{x^2}{2!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x,f) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n}(x,f)$
4. $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2}x^2 + \ldots + \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{1\cdot 2\cdot\ldots\cdot n}x^n + r_n(x,f)$
5. $\ln(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + r_n(x,f) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} + r_n(x,f)$

Примеры

1. $\lim\limits_{x \to 0} \frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = 1$

$$\frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = \frac{x^3/6 + \bar{\bar{o}}(x^4)}{\frac{x^3}{6} + \bar{\bar{o}}(x^3)} = \frac{\frac{1}{6} + x \cdot \frac{\bar{\bar{o}}(x^4)}{x^4}}{\frac{1}{6} + \frac{\bar{\bar{o}}(x^3)}{x^3}} \to 1$$

2. $\lim\limits_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{e}{2}$

$$\frac{(1+x)^{\frac{1}{x}} - e}{x} = \frac{e^{\frac{\ln(1+x)}{x}} - e}{x} = \frac{e^{\frac{1}{x}\left[x - \frac{x^2}{2} + \bar{\bar{o}}(x^2)\right]} - e}{x} =$$

$$= \left(\begin{array}{l} \frac{1}{x}\bar{\bar{o}}(x^2) \text{ есть } \bar{\bar{o}}(x), \\ \text{т.к. } \frac{(1/x)\cdot\bar{\bar{o}}(x^2)}{x} = \frac{\bar{\bar{o}}(x^2)}{x^2} \to 0 \end{array} \right) = \frac{e^{1 - \frac{x}{2} + \bar{\bar{o}}(x)} - e}{x} = \frac{e\left[e^{-\frac{x}{2} + \bar{\bar{o}}(x)} - 1\right]}{x} =$$

$$= \frac{e \cdot \left[1 - \frac{x}{2} + \bar{\bar{o}}(x) + \bar{\bar{o}}\left(-\frac{x}{2} + \bar{\bar{o}}(x)\right) - 1\right]}{x} = \left(\bar{\bar{o}}\left(-\frac{x}{2} + \bar{\bar{o}}(x)\right) = \bar{\bar{o}}\left(-\frac{x}{2}\right) = \bar{\bar{o}}(x)\right)$$

$$= \frac{e \cdot \left[-\frac{x}{2} + \bar{\bar{o}}(x)\right]}{x} = e\left[-\frac{1}{2} + \frac{\bar{\bar{o}}(x)}{x}\right] \to -\frac{e}{2}$$

3. $\lim\limits_{n \to \infty} \left[\frac{\left(1 + \frac{1}{n}\right)^n}{e} \right]^n =$

Т.2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0, $x \in \mathring{U}(x_0)$, $\psi(t) \in C[x_0, x]$ и $\forall t \in (x_0, x)$ $\exists \psi'(t) \neq 0$. $\Rightarrow \exists \xi \in (x_0, x)$:

$$r_n(x,f) = \frac{\psi(x) - \psi(x_0)}{\psi'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (x-\xi)^n$$

$\psi(t) = (x-t)^p$, $p \in \mathbb{N}$ $\quad \psi'(t) = -p(x-t)^{p-1}$

$$r_n(x,f) = \frac{f^{(n+1)}(\xi)}{p \cdot n!}(x-x_0)^p (x-\xi)^{n+1-p}$$ Шлёмильх-Рош

$p = n+1$ $\quad r_n(x,f) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ Лагранж

$p = 1$ $\quad r_n(x,f) = \frac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n$ Коши

Из формы Лагранжа: если $|f^{(n+1)}(x)| \leq M_{n+1} \Rightarrow$

$\Rightarrow |r_n(x,f)| \leq \frac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ — глобальная оценка

Основные формулы

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + r_n(x,f) \qquad x_0 = 0$$

1. $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + r_n(x,f) = \sum_{k=0}^{n} \frac{x^k}{k!} + r_n(x,f)$
2. $\sin x = x - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + r_{2n+1}(x,f) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+1}(x,f)$
3. $\cos x = 1 - \frac{x^2}{2!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n}(x,f) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n}(x,f)$
4. $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2}x^2 + \ldots + \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{1 \cdot 2 \cdot \ldots \cdot n}x^n + r_n(x,f)$
5. $\ln(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + r_n(x,f) = \sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k} + r_n(x,f)$

Примеры

1. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = 1$

$$\frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}} = \frac{x^3/6 + \bar{\bar{o}}(x^4)}{\frac{x^3}{6} + \bar{\bar{o}}(x^3)} = \frac{\frac{1}{6} + x \cdot \frac{\bar{\bar{o}}(x^4)}{x^4}}{\frac{1}{6} + \frac{\bar{\bar{o}}(x^3)}{x^3}} \to 1$$

2. $\lim\limits_{x \to 0} \dfrac{(1+x)^{\frac{1}{x}} - e}{x} = -\dfrac{e}{2}$

$$\frac{(1+x)^{\frac{1}{x}} - e}{x} = \frac{e^{\frac{\ln(1+x)}{x}} - e}{x} = \frac{e^{\frac{1}{x}\left[x - \frac{x^2}{2} + \bar{\bar{o}}(x^2)\right]} - e}{x} =$$

$$= \left(\begin{array}{l} \frac{1}{x}\bar{\bar{o}}(x^2) \text{ есть } \bar{\bar{o}}(x), \\ \text{т.к. } \frac{(1/x)\cdot\bar{\bar{o}}(x^2)}{x} = \frac{\bar{\bar{o}}(x^2)}{x^2} \to 0 \end{array} \right) = \frac{e^{1 - \frac{x}{2} + \bar{\bar{o}}(x)} - e}{x} = \frac{e\left[e^{-\frac{x}{2} + \bar{\bar{o}}(x)} - 1\right]}{x} =$$

$$= \frac{e \cdot \left[1 - \frac{x}{2} + \bar{\bar{o}}(x) + \bar{\bar{o}}\left(-\frac{x}{2} + \bar{\bar{o}}(x)\right) - 1\right]}{x} = \left(\bar{\bar{o}}\left(-\frac{x}{2} + \bar{\bar{o}}(x)\right) = \bar{\bar{o}}\left(-\frac{x}{2}\right) = \bar{\bar{o}}(x)\right)$$

$$= \frac{e \cdot \left[-\frac{x}{2} + \bar{\bar{o}}(x)\right]}{x} = e\left[-\frac{1}{2} + \frac{\bar{\bar{o}}(x)}{x}\right] \to -\frac{e}{2}$$

3. $\lim\limits_{n \to \infty} \left[\dfrac{\left(1 + \frac{1}{n}\right)^n}{e}\right]^n = \dfrac{1}{\sqrt{e}}$

$$\frac{\left(1 + \frac{1}{n}\right)^{n^2}}{e^n} = e^{n^2 \ln\left(1 + \frac{1}{n}\right) - n} = e^{n^2\left[\frac{1}{n} - \frac{1}{2n^2} + \bar{\bar{o}}\left(\frac{1}{n^2}\right)\right] - n} =$$

$$= e^{-\frac{1}{2} + \frac{\bar{\bar{o}}\left(\frac{1}{n^2}\right)}{1/n^2}} \to e^{-\frac{1}{2}}$$

Т.2. $f(x)$: $f^{(n)}(x)$ непр. в окр. x_0, $\exists f^{(n+1)}(x)$ в прок. окр. x_0. $x\in \overset{\circ}{U}(x_0)$, $\psi(t)\in C[x_0,x]$ и $\forall t\in(x_0,x)$ $\exists \psi'(t)\neq 0$. $\Rightarrow \exists \xi\in(x_0,x)$:

$$r_n(x,f)=\frac{\psi(x)-\psi(x_0)}{\psi'(\xi)}\cdot\frac{f^{(n+1)}(\xi)}{n!}\cdot(x-\xi)^n$$

$\psi(t)=(x-t)^p$, $p\in\mathbb{N}$ $\quad \psi'(t)=-p(x-t)^{p-1}$

$$r_n(x,f)=\frac{f^{(n+1)}(\xi)}{p\cdot n!}(x-x_0)^p(x-\xi)^{n+1-p}$$ Шлёмильх – Рош

$p=n+1$ $\quad r_n(x,f)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ Лагранж

$p=1$ $\quad r_n(x,f)=\frac{f^{(n+1)}(\xi)}{n!}(x-x_0)(x-\xi)^n$ Коши

Из формы Лагранжа: если $|f^{(n+1)}(x)|\leqslant M_{n+1}\Rightarrow$

$\Rightarrow |r_n(x,f)|\leqslant \frac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$ – глобальная оценка

Основные формулы

$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x,f)$ $\quad \underline{x_0=0}$

1. $e^x=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+r_n(x,f)=\sum\limits_{k=0}^{n}\frac{x^k}{k!}+r_n(x,f)$
2. $\sin x=x-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+r_{2n+1}(x,f)=\sum\limits_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}+r_{2n+1}(x,f)$
3. $\cos x=1-\frac{x^2}{2!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+r_{2n}(x,f)=\sum\limits_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}+r_{2n}(x,f)$
4. $(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{1\cdot 2}x^2+\ldots+\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{1\cdot 2\cdot\ldots\cdot n}x^n+r_n(x,f)$
5. $\ln(1+x)=x-\frac{x^2}{2}+\ldots+(-1)^{n-1}\frac{x^n}{n}+r_n(x,f)=\sum\limits_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k}+r_n(x,f)$

4. $f(x)=\operatorname{ctg}x-\frac{1}{x}$ $\quad$ Найти главный член вида Cx^α при $x\to 0$.

$$\operatorname{ctg}x-\frac{1}{x}=\frac{\cos x}{\sin x}-\frac{1}{x}=\frac{x\cos x-\sin x}{x\cdot\sin x}=$$

$$=\frac{x\left(1-\frac{x^2}{2}+\bar{\bar{o}}(x^3)\right)-\left(x-\frac{x^3}{6}+\bar{\bar{o}}(x^4)\right)}{x\cdot(x+\bar{\bar{o}}(x^2))}=$$

$$=\frac{x-\frac{x^3}{2}+\bar{\bar{o}}(x^4)-x+\frac{x^3}{6}+\bar{\bar{o}}(x^4)}{x^2+\bar{\bar{o}}(x^3)}=\frac{-\frac{x^3}{3}+\bar{\bar{o}}(x^4)}{x^2+\bar{\bar{o}}(x^3)}\sim$$

$$\sim-\frac{x}{3},\ \text{т.к.}\ \frac{-\frac{x^3}{3}+\bar{\bar{o}}(x^4)}{x^2+\bar{\bar{o}}(x^3)}:\left(-\frac{x}{3}\right)=$$

$$=\frac{x^2+\left(-\frac{3}{x}\right)\cdot\bar{\bar{o}}(x^4)}{x^2+\bar{\bar{o}}(x^3)}=\frac{1+(-3)\cdot x\cdot\frac{\bar{\bar{o}}(x^4)}{x^4}}{1+x\cdot\frac{\bar{\bar{o}}(x^3)}{x^3}}\to 1\ (x\to 0)$$