MATLAB
Linear Programming

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MATLAB Linear Programming

Optimization

Optimization - finding value of a parameter that maximizes or minimizes a function with that parameter

- Talking about mathematical optimization, not optimization of computer code!
- "function" is mathematical function, not MATLAB language function

Optimization

Optimization
- Can have multiple parameters
- Can have multiple functions
  - Parameters can appear linearly or nonlinearly

Linear programming

Linear programming

- Most often used kind of optimization
  - Tremendous number of practical applications
  - "Programming" means determining feasible programs (plans, schedules, allocations) that are optimal with respect to a certain criterion and that obey certain constraints

Linear programming

A *feasible* program is a solution to a linear programming problem and that satisfies certain constraints

In linear programming

- Constraints are linear inequalities
- Criterion is a linear expression
  - Expression called the *objective function*
  - In practice, objective function is often the cost of or profit from some activity

Linear programming

Many important problems in economics and management can be solved by linear programming

Some problems are so common that they're given special names

Linear programming

DIET PROBLEM

You are given a group of foods, their nutritional values and costs. You know how much nutrition a person needs.

What combination of foods can you serve that meets the nutritional needs of a person but costs the least?

Linear programming

BLENDING PROBLEM

– Closely relate to diet problem

Given quantities and qualities of available oils, what is cheapest way to blend them into needed assortment of fuels?

Linear programming

TRANSPORTATION PROBLEM

You are given a group of ports or supply centers of a certain commodity and another group of destinations or markets to which commodity must be shipped. You know how much commodity at each port, how much each market must receive, cost to ship between any port and market.

How much should you ship from each port to each market so as to minimize the total shipping cost?

Linear programming

WAREHOUSE PROBLEM

You are given a warehouse of known capacity and initial stock size. Know purchase and selling price of stock. Interested in transactions over a certain time, e.g., year. Divide time into smaller periods, e.g., months.

How much should you buy and sell each period to maximize your profit, subject to restrictions that

1. Amount of stock at any time can't exceed warehouse capacity
2. You can't sell more stock than you have

Linear programming

Mathematical formulation

The variables $x_1, x_2, \dots x_n$ satisfy the inequalities

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m
\end{aligned}
$$

and $x_1 \ge 0, x_2 \ge 0, \dots x_n \ge 0$. Find the set of values of $x_1, x_2, \dots x_n$ that minimizes (maximizes)

$$x_1 f_1 + x_2 f_2 + \cdots \; x_n f_n$$

Note that a_{pq} and f_i are known

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Mathematical matrix formulation

Find the value of *x* that minimizes (maximizes) $f^T x$ given that $x \geq 0$ and $Ax \leq b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{21} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and } f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

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General procedure

1. Restate problem in terms of equations and inequalities
2. Rewrite in matrix and vector notation
3. Call MATLAB function `linprog` to solve

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Example – diet problem

My son's diet comes from the four basic food groups - chocolate dessert, ice cream, soda, and cheesecake. He checks in a store and finds one of each kind of food, namely, a brownie, chocolate ice cream, Pepsi, and one slice of pineapple cheesecake. Each day he needs at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. Using the table on the next slide that gives the cost and nutrition of each item, figure out how much he should buy and eat of each of the four items he found in the store so that he gets enough nutrition but spends as little (of my money...) as possible.

Linear programming

Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

Linear programming

Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

What are unknowns?

- x_1 = number of brownies to eat each day
- x_2 = number of scoops of chocolate ice cream to eat each day
- x_3 = number of bottles of Coke to drink each day
- x_4 = number of pineapple cheesecake slices to eat each day

In linear programming "unknowns" are called *decision variables*

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Linear programming

Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

Objective is to minimize cost of food. Total daily cost is

Cost = (Cost of brownies) + (Cost of ice cream) + (Cost of Coke) + (Cost of cheesecake)

- Cost of brownies = (Cost/brownie) × (brownies/day) = $2.5x_1$
- Cost of ice cream = x_2
- Cost of Coke = $1.5x_3$
- Cost of cheesecake = $4x_4$

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Linear programming

Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

Therefore, need to minimize

$$2.5x_1 + x_2 + 1.5x_3 + 4x_4$$

Linear programming

Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

Constraint 1 - calorie intake at least 500

- Calories from brownies = (calories/brownie)(brownies/day) = $400x_1$
- Calories from ice cream = $200x_2$
- Calories from Coke = $150x_3$
- Calories from cheesecake = $500x_4$

So constraint 1 is $400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500$

Linear programming

Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

Constraint 2 - chocolate intake at least 6 oz

- Chocolate from brownies = (Chocolate/brownie)(brownies/day) = $3x_1$
- Chocolate from ice cream = $2x_2$
- Chocolate from Coke = $0x_3 = 0$
- Chocolate from cheesecake = $0x_4 = 0$

So constraint 2 is $3x_1 + 2x_2 \geq 6$

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Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

Constraint 3 - sugar intake at least 10 oz

- Sugar from brownies = (sugar/brownie)(brownies/day) = $2x_1$
- Sugar from ice cream = $2x_2$
- Sugar from Coke = $4x_3$
- Sugar from cheesecake = $4x_4$

So constraint 3 is $2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10$

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Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

Constraint 4 - fat intake at least 8 oz

- Fat from brownies = (fat/brownie)(brownies/day) = $2x_1$
- Fat from ice cream = $4x_2$
- Fat from Coke = $1x_3$
- Fat from cheesecake = $5x_4$

So constraint 4 is $2x_1 + 4x_2 + x_3 + 5x_4 \geq 8$

Linear programming

Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

Finally, we assume that the amounts eaten are non-negative, i.e., we ignore throwing up. This means that we have

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0, \text{ and } x_4 \geq 0$$

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Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

Putting it all together, we have to minimize $2.5x_1 + x_2 + 1.5x_3 + 4x_4$ subject to the constraints

$$\begin{aligned} 400x_1 + 200x_2 + 150x_3 + 500x_4 &\ge 500 \\ 3x_1 + 2x_2 &\ge 6 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 &\ge 10 \\ 2x_1 + 4x_2 + x_3 + 5x_4 &\ge 8 \end{aligned}$$

and

$$\begin{aligned} x_1 &\ge 0 \\ x_2 &\ge 0 \\ x_3 &\ge 0 \\ x_4 &\ge 0 \end{aligned}$$

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Linear programming

Example – diet problem

Food	Calories	Chocolate	Sugar (ounces)	Fat (ounces)	Cost (serving)
Brownie	400	3	2	2	$2.50 / brownie
Chocolate ice cream	200	2	2	4	$1.00 / scoop
Coke	150	0	4	1	$1.50 / bottle
Pineapple cheesecake	500	0	4	5	$4.00 / slice

In matrix notation, want to

$$\text{minimize } f^T x \text{ subject to } Ax \geq b \text{ and } x \geq 0$$

where

$$A = \begin{bmatrix} 400 & 200 & 150 & 500 \\ 3 & 2 & 0 & 0 \\ 2 & 2 & 4 & 4 \\ 2 & 4 & 1 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 500 \\ 6 \\ 10 \\ 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \text{and } f = \begin{bmatrix} 2.5 \\ 1 \\ 1.5 \\ 4 \end{bmatrix}$$

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Linear programming

MATLAB solves linear programming problem

$$\text{minimize } f^T x \text{ such that } \begin{cases} A \cdot x \le b \\ Aeq \cdot x = beq \\ lb \le x \le ub \end{cases}$$

where *x, b, beq, lb,* and *ub* are vectors and *A* and *Aeq* are matrices.

- Can use one or more of the constraints
- "lb" means "lower bound", "ub" means "upper bound"
  - Often have *lb = 0* and *ub* = ∞, i.e., no upper bound

Linear programming

MATLAB linear programming solver is `linprog()`, which you can call various ways:

```
x = linprog(f,A,b)
x = linprog(f,A,b,Aeq,beq)
x = linprog(f,A,b,Aeq,beq,lb,ub)
x = linprog(f,A,b,Aeq,beq,lb,ub,x0)
x = linprog(f,A,b,Aeq,beq,lb,ub,x0,options)
x = linprog(problem)
[x,fval] = linprog(...)
[x,fval,exitflag] = linprog(...)
[x,fval,exitflag,output] = linprog(...)
[x,fval,exitflag,output,lambda] = linprog(...)
```

Linear programming

Example – diet problem

Us: minimize $f^T x$ subject to $Ax \geq b$ and $0 \leq x$

$$A = \begin{bmatrix} 400 & 200 & 150 & 500 \\ 3 & 2 & 0 & 0 \\ 2 & 2 & 4 & 4 \\ 2 & 4 & 1 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 500 \\ 6 \\ 10 \\ 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \text{and } f = \begin{bmatrix} 2.5 \\ 1 \\ 1.5 \\ 4 \end{bmatrix}$$

MATLAB:

$$\text{minimize } f^T x \text{ such that } \begin{cases} A \cdot x \leq b \\ Aeq \cdot x = beq \\ lb \leq x \leq ub \end{cases}$$

Note two differences:

Linear programming

Example – diet problem

ISSUE 1 - We have $Ax \geq b$ but need $Ax \leq b$

One way to handle is to note that

if $Ax \geq b$ then $-Ax \leq -b$, so can have MATLAB use constraint $(-A)x \leq (-b)$

ISSUE 2 - We have $0 \leq x$ but MATLAB wants $lb \leq x \leq ub$. Handle by omitting *ub* in call of `linprog()`. If omitted, MATLAB assumes no upper bound

Linear programming

Example – diet problem

`x = linprog(f,A,b,Aeq,beq,lb,ub)`

- We'll actually call

  `x = linprog(f,A,b,Aeq,beq,lb)`

- If don't have equality constraints, pass `[]` for `Aeq` and `beq`

Linear programming

Example – diet problem

Follow along now

```
>> A = -[ 400 200 150 500; 3 2 0 0; 2 2 4 4;...
          2 4 1 5 ];
>> b = -[ 500 6 10 8 ]';
>> f = [ 2.5 1 1.5 4]';
>> lb = [ 0 0 0 0 ]';
>> x = linprog( f, A, b, [], [], lb )
       Optimization terminated.
       x = 0.0000 % brownies
           3.0000 % chocolate ice cream
           1.0000 % Coke
           0.0000 % cheesecake
```

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Linear programming

Example – diet problem

Optimal solution is $x = [\,0\ 3\ 1\ 0\,]^T$. In words, my son should eat 3 scoops of ice cream and drink 1 Coke each day.

Linear programming

Example – diet problem

A constraint is *binding* if both sides of the constraint inequality are equal when the optimal solution is substituted.

For $x = [\,0\ 3\ 1\ 0\,]^T$ the set

$$\begin{aligned} 400x_1 + 200x_2 + 150x_3 + 500x_4 &\geq 500 \\ 3x_1 + 2x_2 &\geq 6 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 &\geq 10 \\ 2x_1 + 4x_2 + x_3 + 5x_4 &\geq 8 \end{aligned}$$

becomes

$$\begin{aligned} 750 &\geq 500 \\ 6 &\geq 6 \\ 10 &\geq 10 \\ 13 &\geq 8 \end{aligned}$$

,

so the chocolate and sugar constraints are binding. The other two are *nonbinding*

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Linear programming

Example – diet problem

How many calories, and how much chocolate, sugar and fat will he get each day?

```
>> -A*x
ans = 750.0000 % calories
        6.0000 % chocolate
       10.0000 % sugar
       13.0000 % fat
```

How much money will this cost?

```
>> f'*x
ans = 4.5000 % dollars
```


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Example – diet problem

Because it's common to want to know the value of the objective function at the optimum,

`linprog()` can return that to you

`[x fval] = linprog(f,A,b,Aeq,beq,lb,ub)`

where $fval = f^T x$

```
>> [x fval] = linprog( f, A, b, [], [], lb )
x = 0.0000
    3.0000
    1.0000
    0.0000
fval = 4.5000
```

Linear programming

Special kinds of solutions

Usually a linear programming problem has a unique (single) optimal solution. However, there can also be:

1. No feasible solutions
2. An unbounded solution. There are solutions that make the objective function arbitrarily large (max problem) or arbitrarily small (min problem)
3. An infinite number of optimal solutions. The technique of *goal programming* is often used to choose among alternative optimal solutions. (Won't consider this case more)

Linear programming

Can tell about the solution MATLAB finds by using third output variable:

```
[x fval exitflag] =...
linprog(f,A,b,Aeq,beq,lb,ub)
```

`exitflag` - integer identifying the reason the algorithm terminated. Values are

1	Function converged to a solution x.
0	Number of iterations exceeded options.
-2	No feasible point was found.
-3	Problem is unbounded.
-4	NaN value was encountered during execution of the algorithm.
-5	Both primal and dual problems are infeasible.
-7	Search direction became too small. No further progress could be made.

Linear programming

Try It

Solve the following problem and display the optimal solution, the value of the objective value there, and the exit flag from `linprog()`

Maximize $z = 2x_1 - x_2$ subject to

$$\begin{aligned} x_1 - x_2 &\le 1 \\ 2x_1 + x_2 &\ge 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Linear programming

Try It

First multiply second equation by -1 to get

$$\begin{aligned} x_1 - x_2 &\le 1 \\ -2x_1 - x_2 &\le -6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Then, with objective function $z = 2x_1 - x_2$ rewrite in matrix form:

$$A = \begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -6 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ and } lb = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Try It

Linear programming

$$A = \begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -6 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, f = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ and } lb = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

```
>> A = [ 1 -1; -2 -1 ];
>> b = [ 1 -6 ]';
>> f = [ 2 -1 ]';
>> lb = [ 0 0 ]';
```

Linear programming

Try It

IMPORTANT - `linprog()` tries to minimize the objective function. If you want to maximize the objective function, pass `-f` and use `-fval` as the maximum value of the objective function

Linear programming

Try It

```
>> [x fval exitflag] = linprog( -f, A, b, [],[], lb )
Exiting: One or more of the residuals, duality gap,
or total relative error has grown 100000 times
greater than its minimum value so far: the dual
appears to be infeasible (and the primal unbounded).
(The primal residual < TolFun=1.00e-008.)
x = 1.0e+061 *
    4.4649
    4.4649
fval = -4.4649e+061 (-fval = 4.4649e+061 !!!)
exitflag = -3 (Problem is unbounded)
```

Linear programming

Try It

A farmer has 10 acres to plant in wheat and rye. He has to plant at least 7 acres. However, he has only \$1200 to spend and each acre of wheat costs \$200 to plant and each acre of rye costs \$100 to plant. Moreover, the farmer has to get the planting done in 12 hours and it takes an hour to plant an acre of wheat and 2 hours to plant an acre of rye. If the profit is \$500 per acre of wheat and \$300 per acre of rye how many acres of each should be planted to maximize profits?

Linear programming

Try It

Decision variables

- x is number of acres of wheat to plant
- y is number of acres of rye to plant

Constraints

- "has 10 acres to plant in wheat and rye"
  - In math this is $x + y \leq 10$
- " has to plant at least 7 acres"
  - In math this is $x + y \geq 7$

Linear programming

Try It

Constraints

- "he has only $1200 to spend and each acre of wheat costs $200 to plant and each acre of rye costs $100 to plant"
  - In math this is $200x + 100y \leq 1200$

Linear programming

Try It

Constraints

- "the farmer has to get the planting done in 12 hours and it takes an hour to plant an acre of wheat and 2 hours to plant an acre of rye "
  - In math this is $1x + 2y \leq 12$

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Try It

Objective function

- "... the profit is \$500 per acre of wheat and \$300 per acre of rye"
  - In math this is $z = 500x + 300y$

Linear programming

Try It

Put it together

- Constraints:

$$
\begin{aligned}
x + y &\le 10 \\
x + y &\ge 7 \\
200x + 100y &\le 1200 \\
x + 2y &\le 12 \\
x &\ge 0 \\
y &\ge 0
\end{aligned}
$$

- Objective function:

$$z = 500x + 300y$$

Linear programming

Try It

Rename x to x_1 and y to x_2

Change $x + y \geq 7$ to $-x - y \leq -7$ and then to $-x_1 - x_2 \leq -7$

$$
\begin{aligned}
x_1 + x_2 &\leq 10 \\
-x_1 - x_2 &\leq -7 \\
200x_1 + 100x_2 &\leq 1200 \\
x_1 + 2x_2 &\leq 12 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}
$$

$$z = 500x_1 + 300x_2$$

Linear programming

Try It

Write in matrix form

Maximize $z = 500x_1 + 300x_2$

$$\begin{aligned} x_1 + x_2 &\le 10 \\ -x_1 - x_2 &\le -7 \\ 200x_1 + 100x_2 &\le 1200 \\ x_1 + 2x_2 &\le 12 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 200 & 100 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ -7 \\ 1200 \\ 12 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, f = \begin{bmatrix} 500 \\ 300 \end{bmatrix} \text{ and } lb = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Maximize $z = f^T x$

Linear programming

Try It

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 200 & 100 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ -7 \\ 1200 \\ 12 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, f = \begin{bmatrix} 500 \\ 300 \end{bmatrix} \text{ and } lb = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$z = f^T x$$

Find solution that maximizes profit. Display both

```
>> A = [ 1 1; -1 -1; 100 200; 2 1];
>> b = [ 10 -7 1200 12 ]';
>> f = [ 300 500 ]';
>> lb = [ 0 0 ]';
>> [x fval] = linprog( -f, A, b, [], [], lb );
>> x'
ans = 4.0000    4.0000
>> maxProfit = -fval
maxProfit =  3.2000e+003
```

Linear programming

Try It – blending problem

Alloy Mixture Optimization (minimize expenses)

There are four metals with the following properties:

Metal	Density	%Carbon	%Phosphor	Price ($/kg)
A	6500	0.2	0.05	2.0
B	5800	0.35	0.015	2.5
C	6200	0.15	0.065	1.5
D	5900	0.11	0.1	2.0

We want to make an alloy with properties in the following range:

Range	Density	%Carbon	%Phosphor
Minimum	5950	0.1	0.045
Maximum	6050	0.3	0.055

What mixture of metals should we use to minimize the cost of the alloy?

Linear programming

Try It – blending problem

Decision variables

- x_1 is fraction of total alloy that is metal A
- x_2 is fraction of total alloy that is metal B
- x_3 is fraction of total alloy that is metal C
- x_4 is fraction of total alloy that is metal D

Linear programming

Metal	Density	%Carbon	%Phosphor	Price ($/kg)
A	6500	0.2	0.05	2.0
B	5800	0.35	0.015	2.5
C	6200	0.15	0.065	1.5
D	5900	0.11	0.1	2.0

Range	Density	%Carbon	%Phosphor
Minimum	5950	0.1	0.045
Maximum	6050	0.3	0.055

Try It – blending problem

Density constraints

- Alloy density must be at least 5950
  - In math this is $6500x_1 + 5800x_2 + 6200x_3 + 5900x_4 \geq 5950$
- Alloy density must be at most 6050
  - In math this is $6500x_1 + 5800x_2 + 6200x_3 + 5900x_4 \leq 6050$

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Metal	Density	%Carbon	%Phosphor	Price ($/kg)
A	6500	0.2	0.05	2.0
B	5800	0.35	0.015	2.5
C	6200	0.15	0.065	1.5
D	5900	0.11	0.1	2.0

Range	Density	%Carbon	%Phosphor
Minimum	5950	0.1	0.045
Maximum	6050	0.3	0.055

Try It – blending problem

Carbon constraints

- Carbon concentration must be at least 0.1
  - In math this is $0.2x_1 + 0.35x_2 + 0.15x_3 + 0.11x_4 \geq 0.1$
- Carbon concentration must be at most 0.3
  - In math this is $0.2x_1 + 0.35x_2 + 0.15x_3 + 0.11x_4 \leq 0.3$

Linear programming

Metal	Density	%Carbon	%Phosphor	Price ($/kg)
A	6500	0.2	0.05	2.0
B	5800	0.35	0.015	2.5
C	6200	0.15	0.065	1.5
D	5900	0.11	0.1	2.0

Range	Density	%Carbon	%Phosphor
Minimum	5950	0.1	0.045
Maximum	6050	0.3	0.055

Try It – blending problem

Phosphor constraints

- Phosphor concentration must be at least 0.1
  - In math this is $0.05x_1 + 0.015x_2 + 0.065x_3 + 0.1x_4 \geq 0.045$
- Phosphor concentration must be at most 0.3
  - In math this is $0.05x_1 + 0.015x_2 + 0.065x_3 + 0.1x_4 \leq 0.055$

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Linear programming

Try It – blending problem

Constraints

Since only the four metals will make up the alloy, the sum of the fractional amounts must be one:

$$x_1 + x_2 + x_3 + x_4 = 1$$

Fractional parts must be non-negative:

$$0 \le x_i \quad \text{for} \quad i = 1,2,3,4$$

(Each part must also be ≤ 1, but that's handled by first equation.)

Linear programming

Metal	Density	%Carbon	%Phosphor	Price ($/kg)
A	6500	0.2	0.05	2.0
B	5800	0.35	0.015	2.5
C	6200	0.15	0.065	1.5
D	5900	0.11	0.1	2.0

Try It – blending problem

Objective function

Cost per kg $z = 2.0x_1 + 2.5x_2 + 1.5x_3 + 2.0x_4$

Linear programming

Try It – blending problem

Put it together

– Constraints:

(Convert ≥ to ≤)

$$
\begin{aligned}
-\left(6500x_1 + 5800x_2 + 6200x_3 + 5900x_4\right) &\le -5950 \\
6500x_1 + 5800x_2 + 6200x_3 + 5900x_4 &\le 6050 \\
-\left(0.2x_1 + 0.35x_2 + 0.15x_3 + 0.11x_4\right) &\le -0.1 \\
0.2x_1 + 0.35x_2 + 0.15x_3 + 0.11x_4 &\le 0.3 \\
-\left(0.05x_1 + 0.015x_2 + 0.065x_3 + 0.1x_4\right) &\le -0.045 \\
0.05x_1 + 0.015x_2 + 0.065x_3 + 0.1x_4 &\le 0.055 \\
x_1 + x_2 + x_3 + x_4 &= 1 \\
x_i \ge 0, \quad i &= 1,2,3,4
\end{aligned}
$$

–Objective function:

$$z = 2.0x_1 + 2.5x_2 + 1.5x_3 + 2.0x_4$$

Linear programming

Try It – blending problem

Write in matrix form

$$A=\begin{bmatrix}-6500 & -5800 & -6200 & -5900\\ 6500 & 5800 & 6200 & 5900\\ -0.2 & -0.35 & -0.15 & -0.11\\ 0.2 & 0.35 & 0.15 & 0.11\\ -0.05 & -0.015 & -0.065 & -0.1\\ 0.05 & 0.015 & 0.065 & 0.1\end{bmatrix},\quad b=\begin{bmatrix}-5950\\ 6050\\ -0.1\\ 0.3\\ -0.045\\ 0.055\end{bmatrix},\quad x=\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

$$f=\begin{bmatrix}2.0\\ 2.5\\ 1.5\\ 2.0\end{bmatrix},\ \mathrm{A_{EQ}}=\begin{bmatrix}1\\ 1\\ 1\\ 1\end{bmatrix}^{\mathrm{T}},\ \mathrm{b_{EQ}}=1,\text{ and } lb=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

$$\begin{aligned}
-(6500x_1+5800x_2+6200x_3+5900x_4)&\le -5950\\
6500x_1+5800x_2+6200x_3+5900x_4&\le 6050\\
-(0.2x_1+0.35x_2+0.15x_3+0.11x_4)&\le -0.1\\
0.2x_1+0.35x_2+0.15x_3+0.11x_4&\le 0.3\\
-(0.05x_1+0.015x_2+0.065x_3+0.1x_4)&\le -0.045\\
0.05x_1+0.015x_2+0.065x_3+0.1x_4&\le 0.055\\
x_1+x_2+x_3+x_4&=1\\
x_i\ge 0,\quad i&=1,2,3,4
\end{aligned}$$

Minimize $z=f^T x$

Linear programming

$$A=\begin{bmatrix}-6500 & -5800 & -6200 & -5900\\ 6500 & 5800 & 6200 & 5900\\ -0.2 & -0.35 & -0.15 & -0.11\\ 0.2 & 0.35 & 0.15 & 0.11\\ -0.05 & -0.015 & -0.065 & -0.1\\ 0.05 & 0.015 & 0.065 & 0.1\end{bmatrix},\quad b=\begin{bmatrix}-5950\\ 6050\\ -0.1\\ 0.3\\ -0.045\\ 0.055\end{bmatrix},\quad x=\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

$$f=\begin{bmatrix}2.0\\ 2.5\\ 1.5\\ 2.0\end{bmatrix},\ \mathrm{A}_{\mathrm{EQ}}=\begin{bmatrix}1\\ 1\\ 1\\ 1\end{bmatrix}^{\mathrm{T}},\ \mathrm{b}_{\mathrm{EQ}}=1,\ \text{and}\ lb=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

Try It – blending problem

```
>> A = [-6500 -5800 -6200 -5900; 6500 5800 6200 5900;...
         -0.2 -0.35 -0.15 -0.11; 0.2 0.35 0.15 0.11;...
        -0.05 -0.015 -0.065 -0.1; 0.05 0.015 0.065 0.1 ];
>> b = [ -5950 6050 -0.1 0.3 -0.045 0.055 ]';
>> f = [ 2 2.5 1.5 2 ]';
>> Aeq = [ 1 1 1 1 ];
>> beq = 1;
>> lb = [ 0 0 0 0 ]';
```

Linear programming

Try It – blending problem

```
>> [x fval] = linprog( f, A, b, Aeq, beq, lb )
Optimization terminated.
x = 0.0000 <- Metal A
    0.2845 <- Metal B
    0.5948 <- Metal C
    0.1207 <- Metal D
fval = 1.8448 <- Profit in $/kg
```

MATLAB Linear Programming

Questions?

The End