Элементы гидродинамики

и теплопереноса

в гелии II

Гелий II - квантовая жидкость

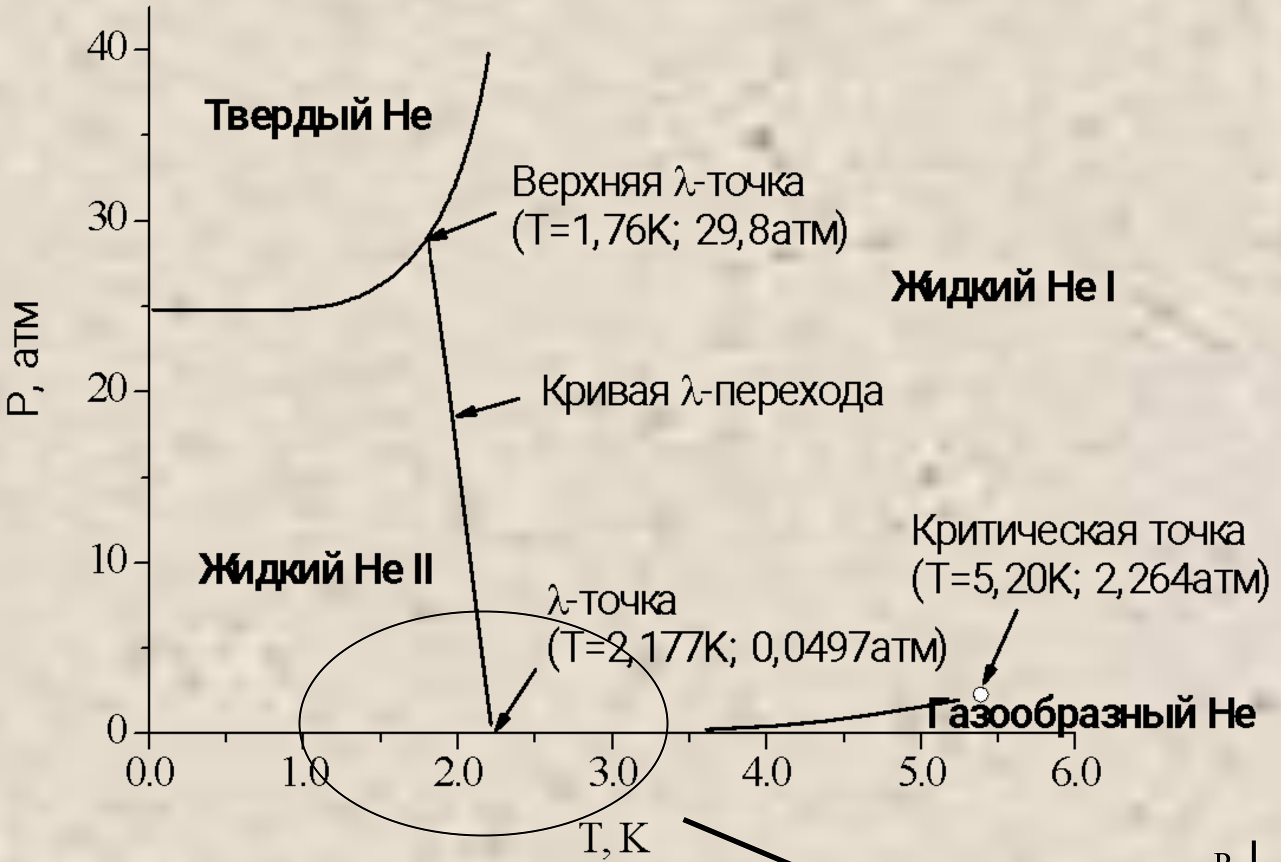
Соотношение неопределенностей Гейзенберга:

$$\Delta p \cdot \Delta x \geq \hbar \quad (1)$$

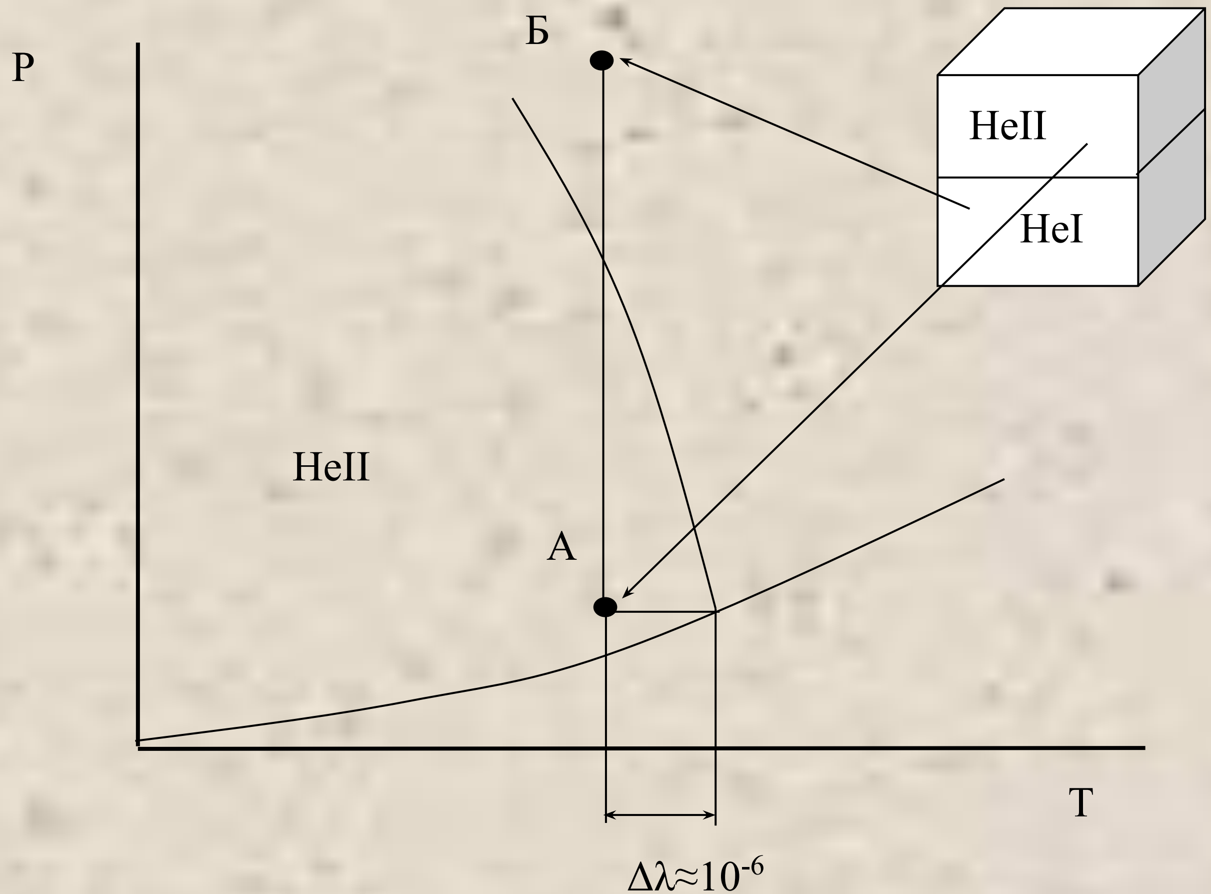
Неопределенность (флуктуации) энергии атома:

$$\Delta E \geq \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{2m(\Delta x)^2} \quad (2)$$

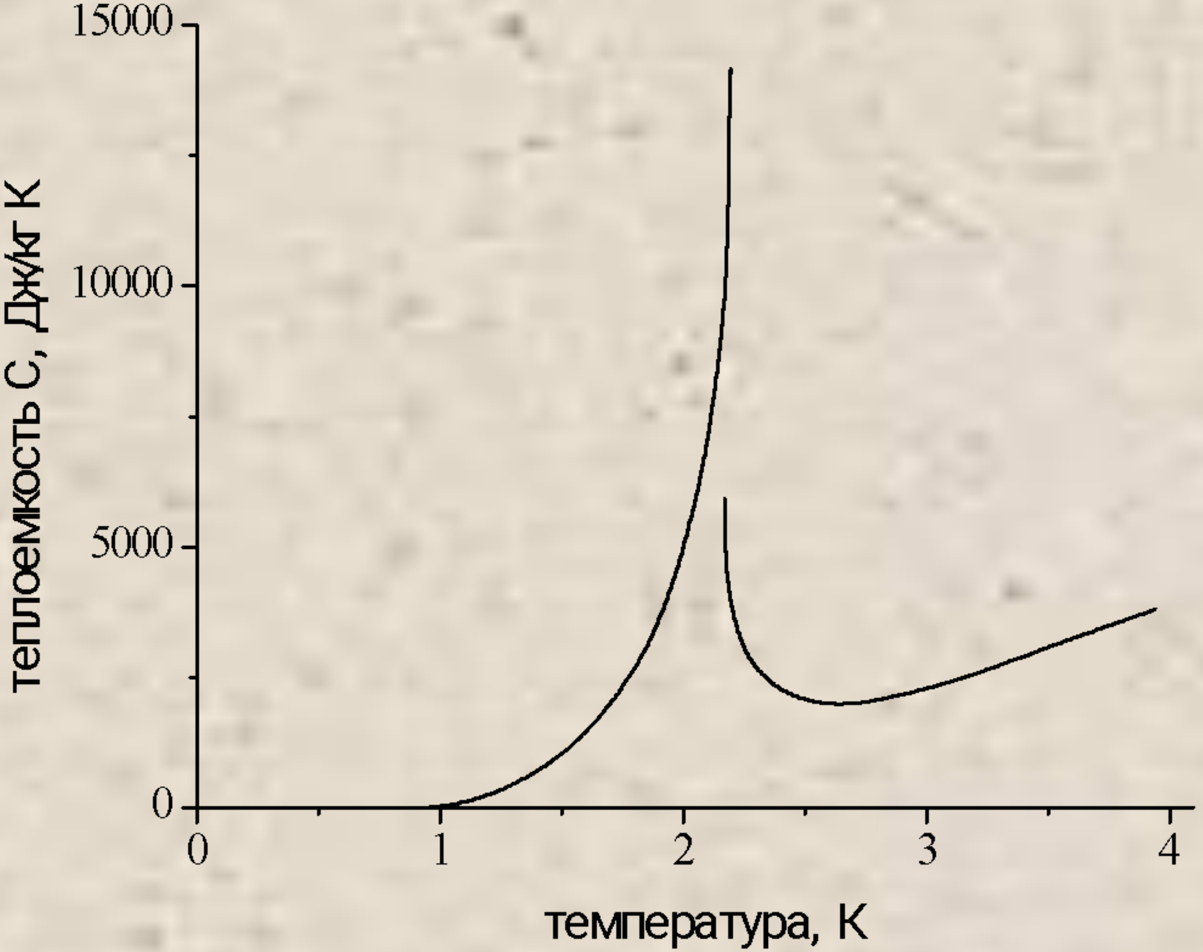
Р–Т диаграмма



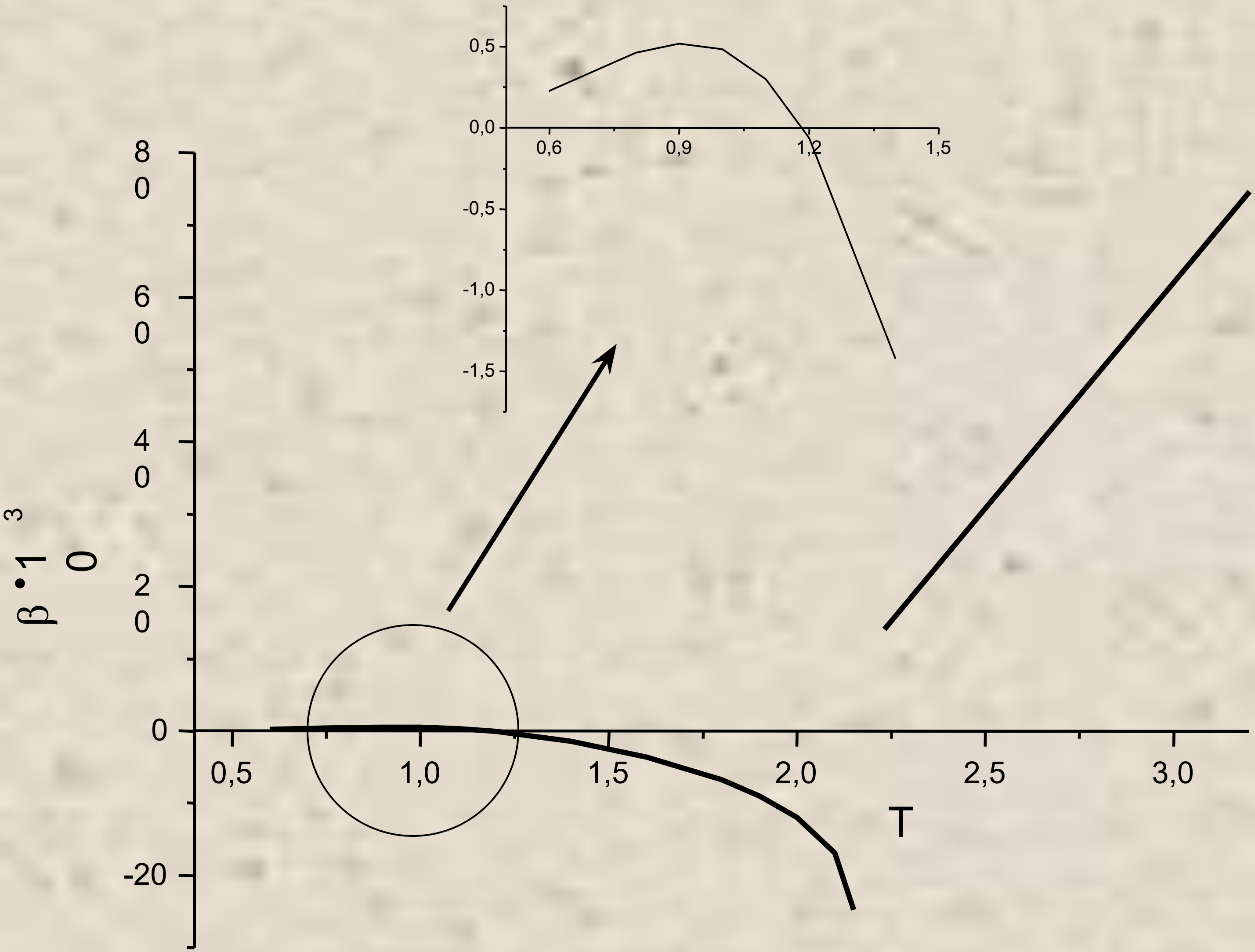
P–T диаграмма



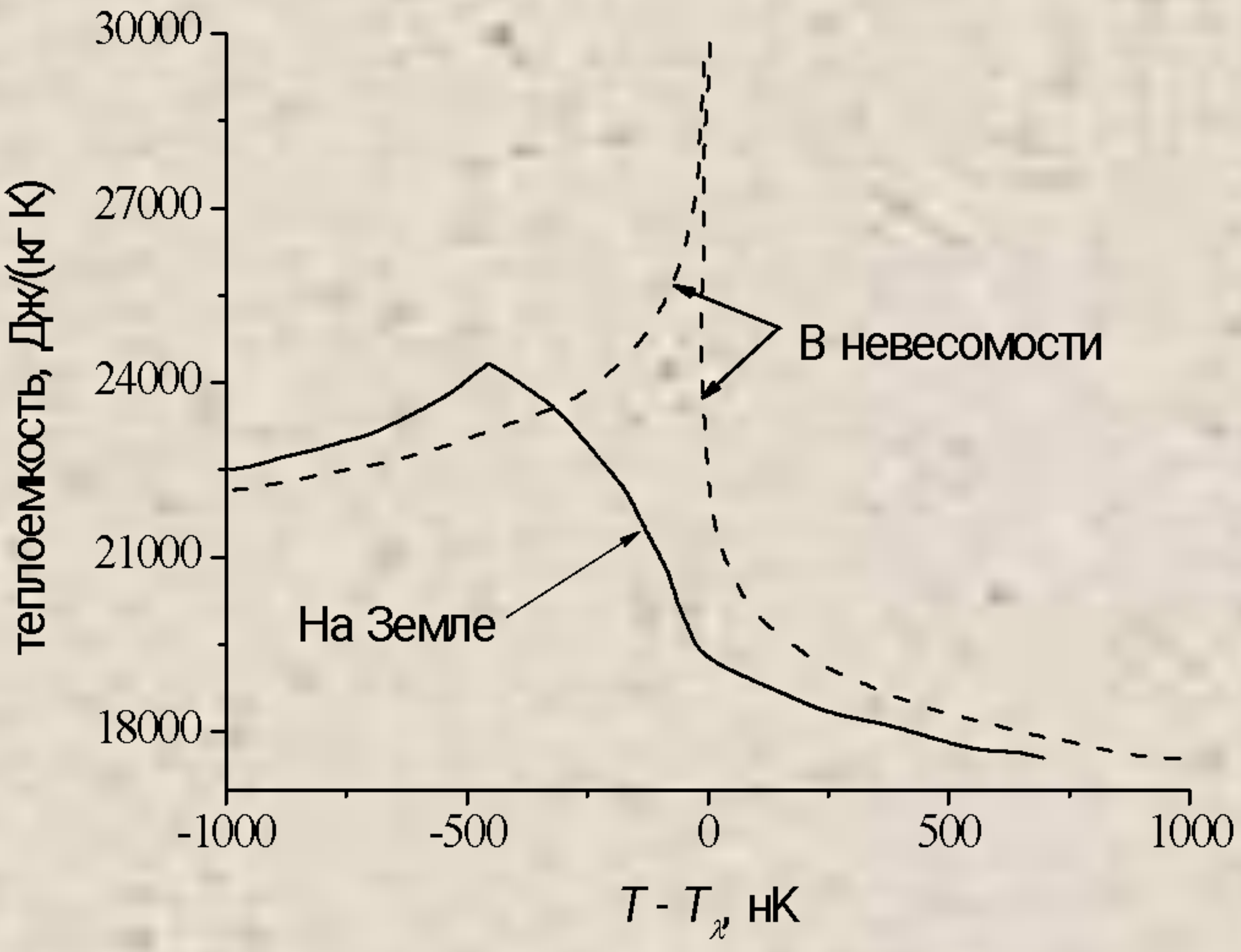
Теплоемкость вблизи лямбда-точки



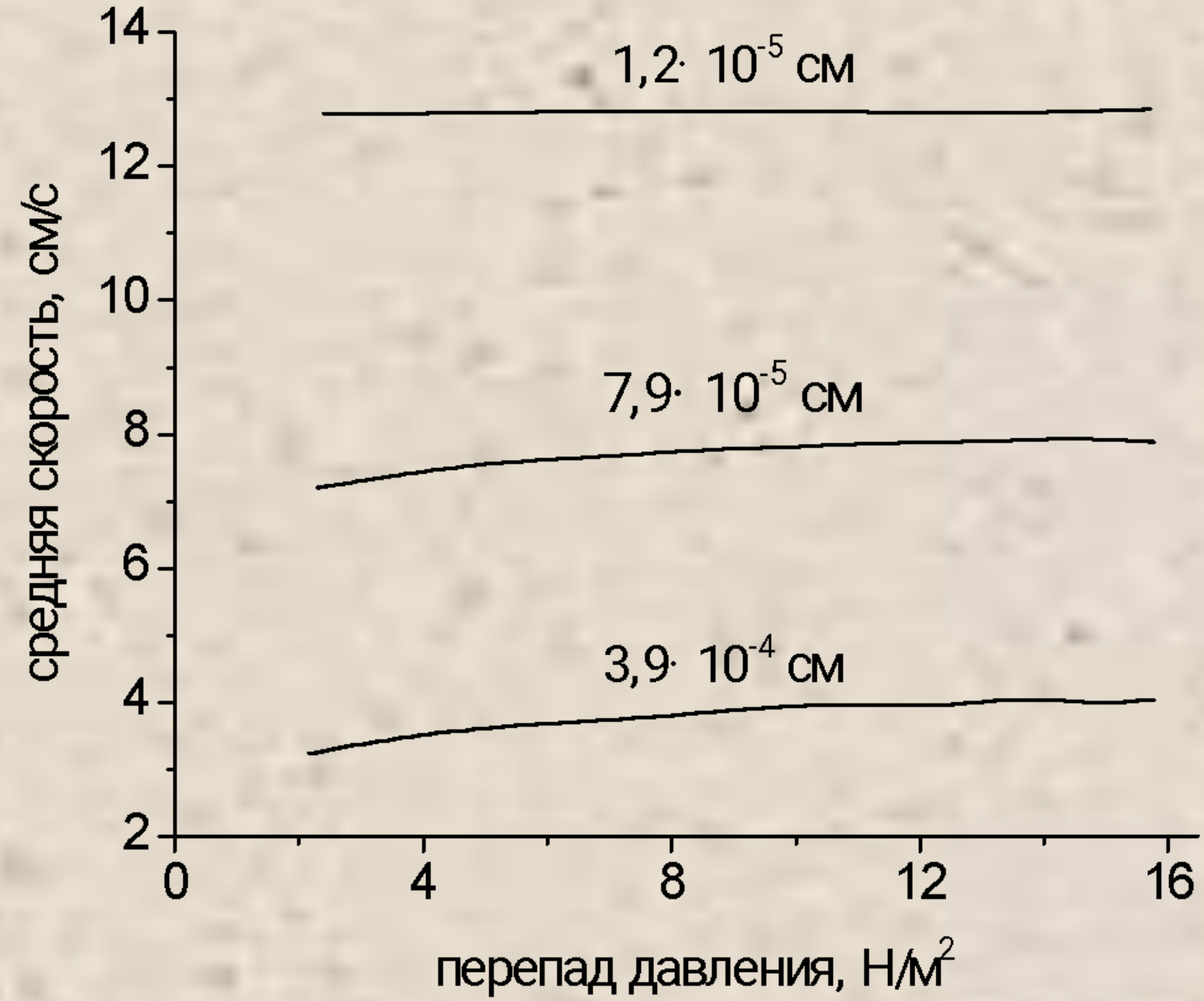
Температурный коэффициент объемного расширения



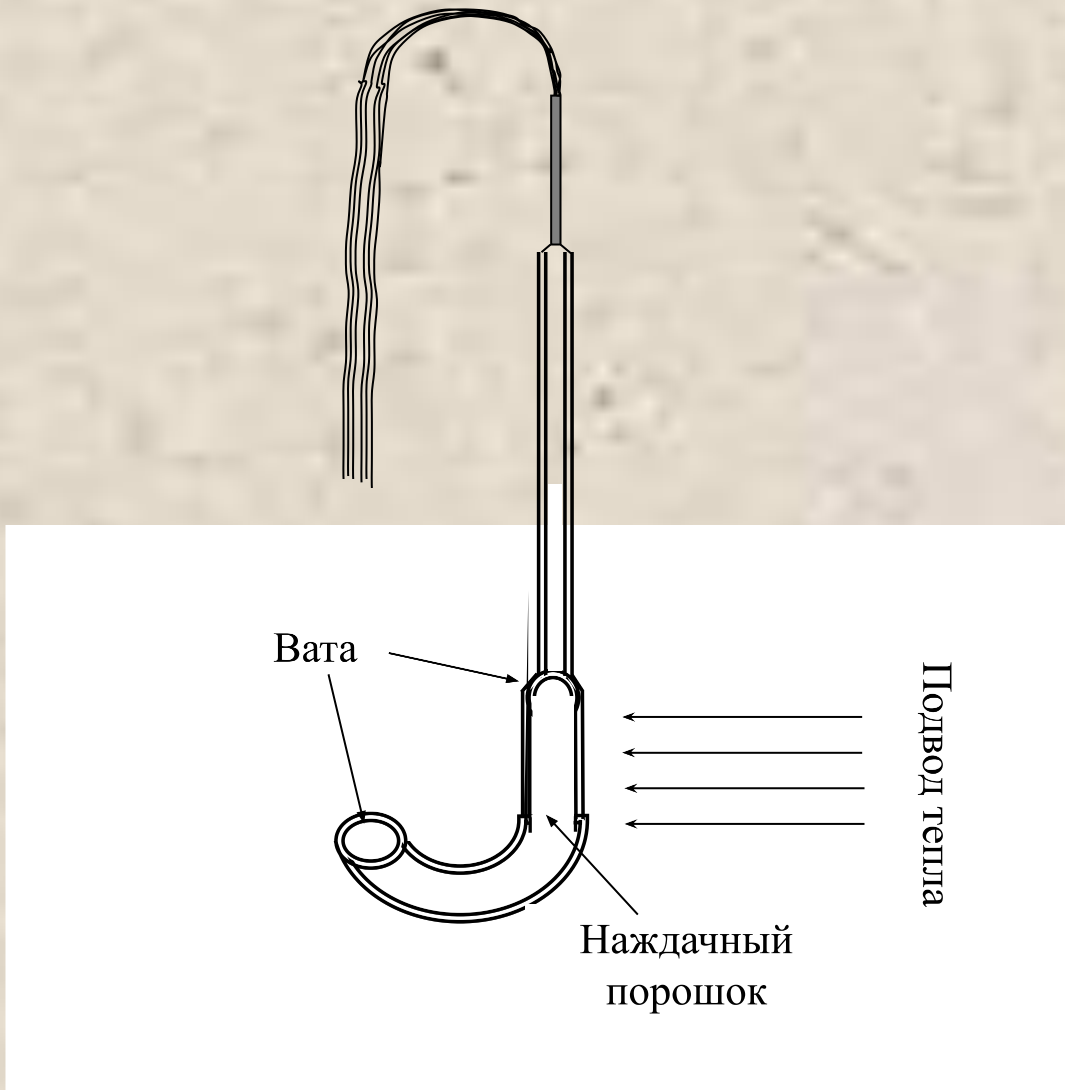
Теплоемкость вблизи лямбда-точки



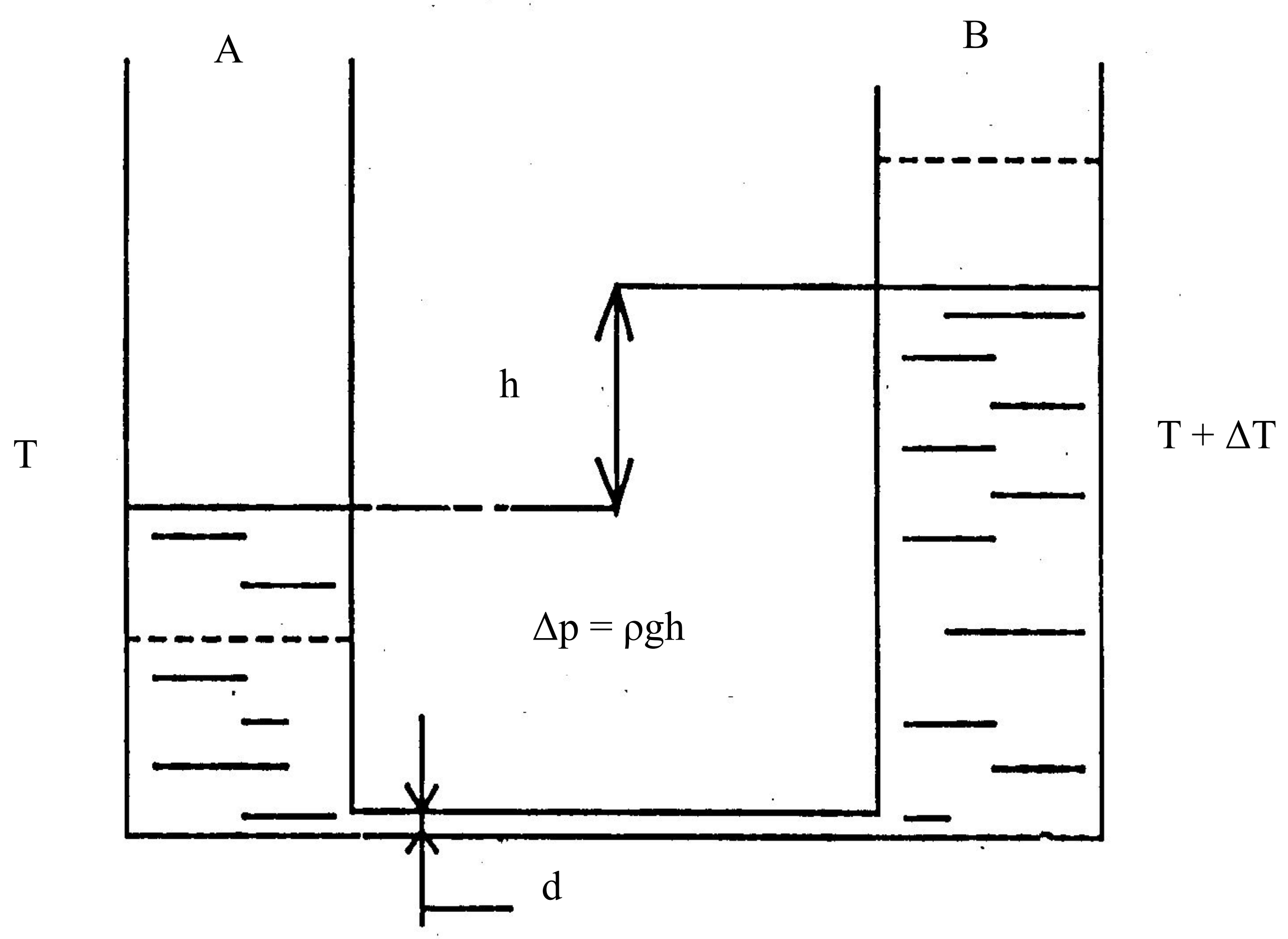
Скорость течения гелия II



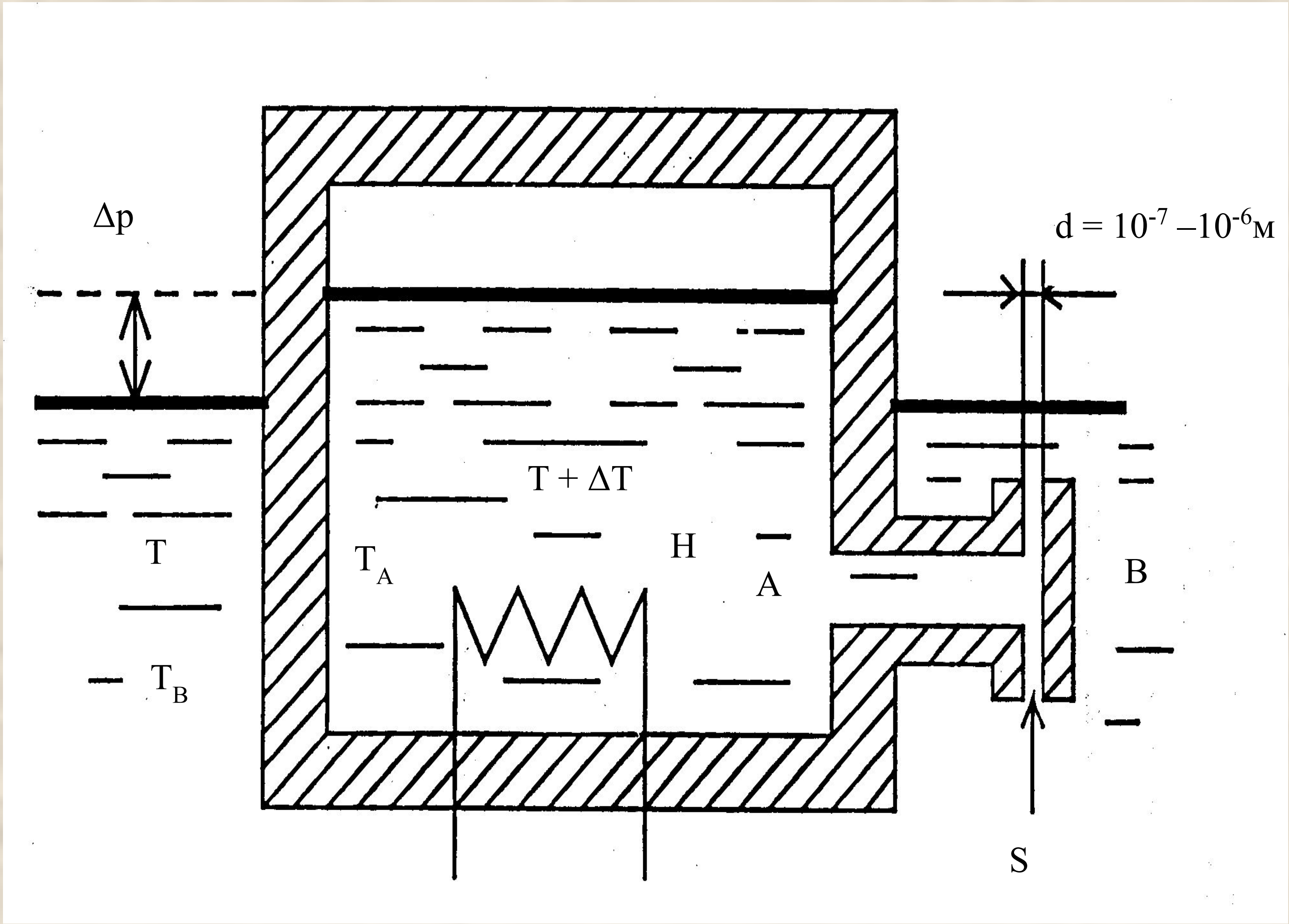
Эффект фонтанирования



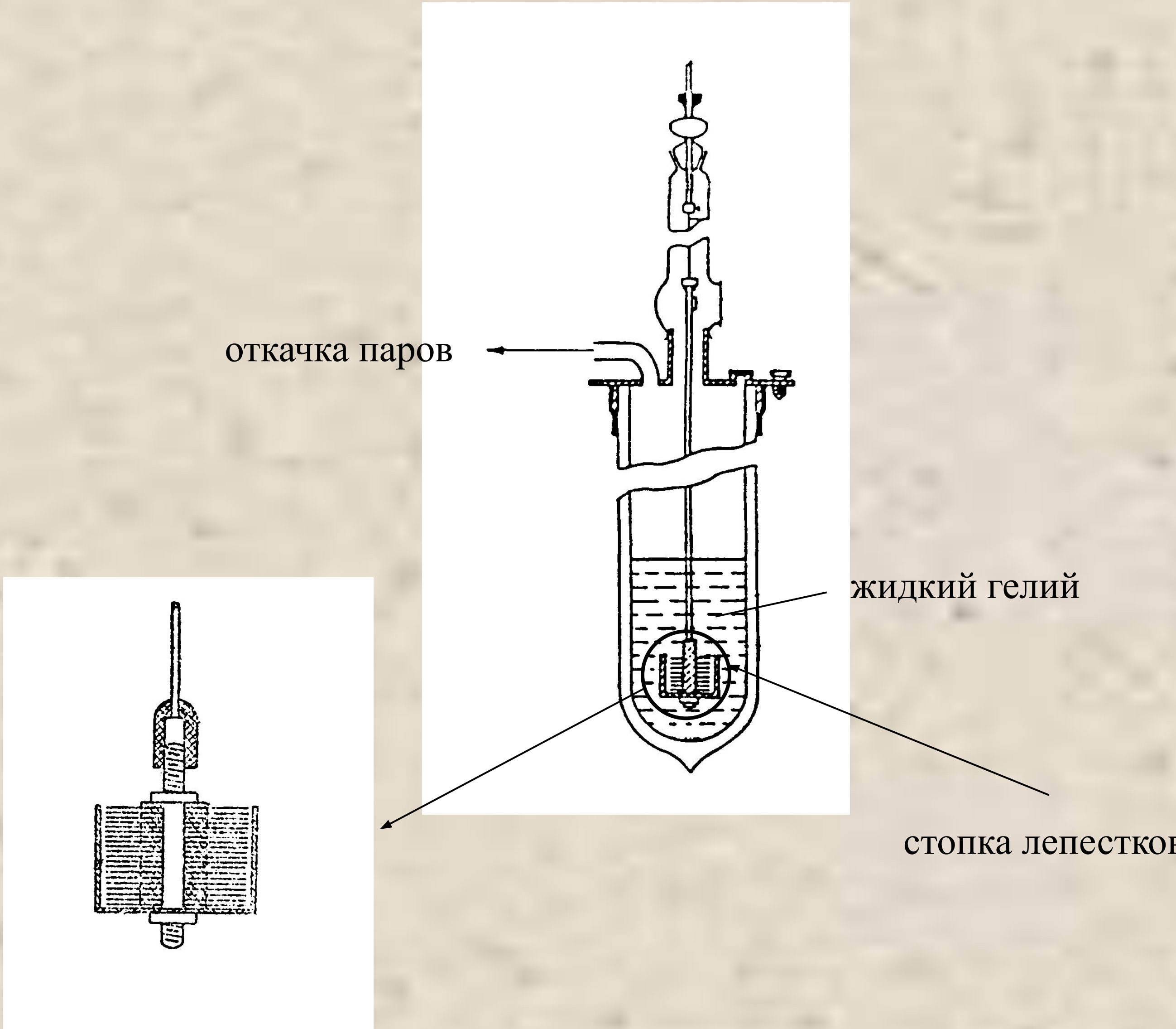
Механокалорический эффект



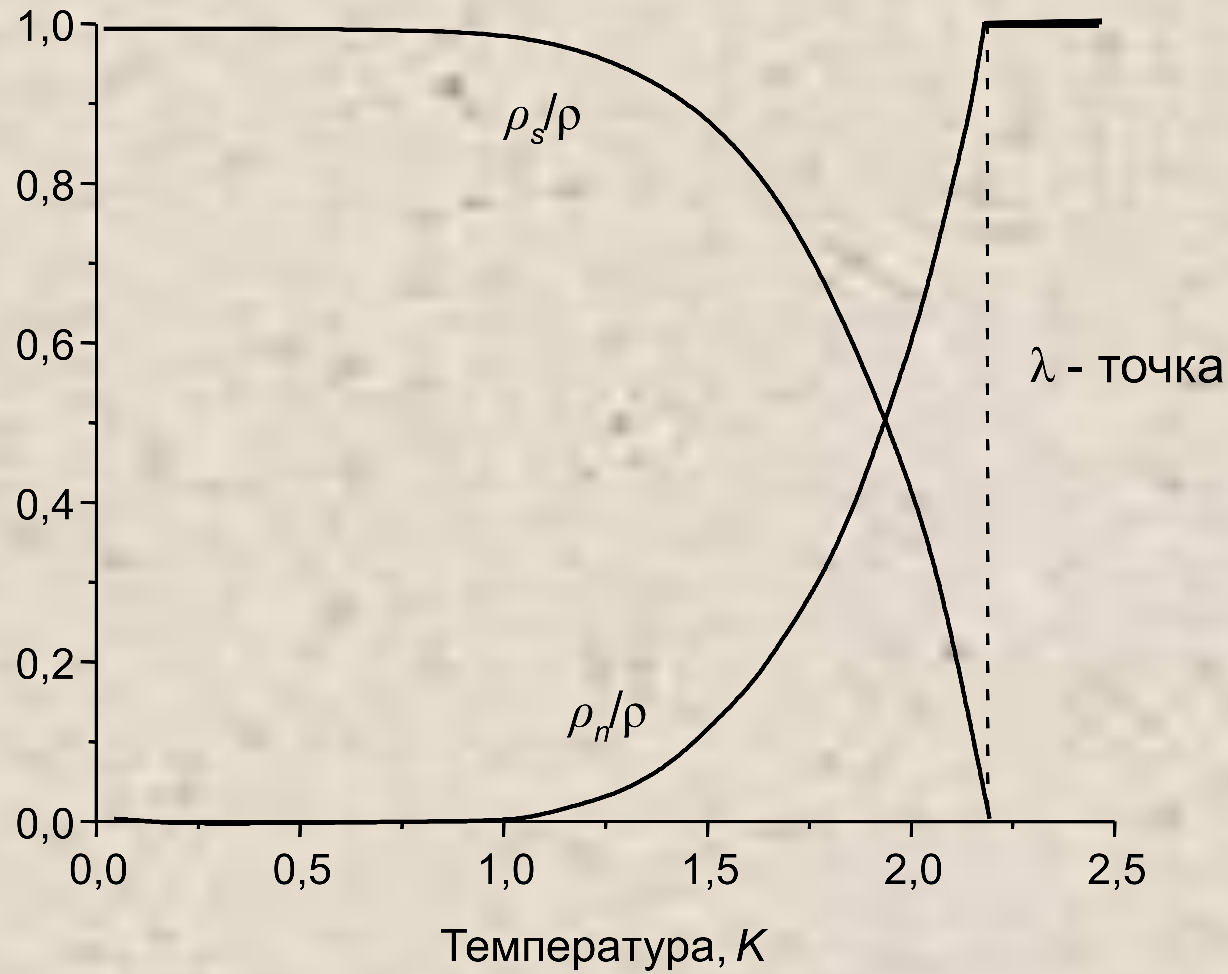
Термомеханический эффект. Эксперимент П.Л. Капицы



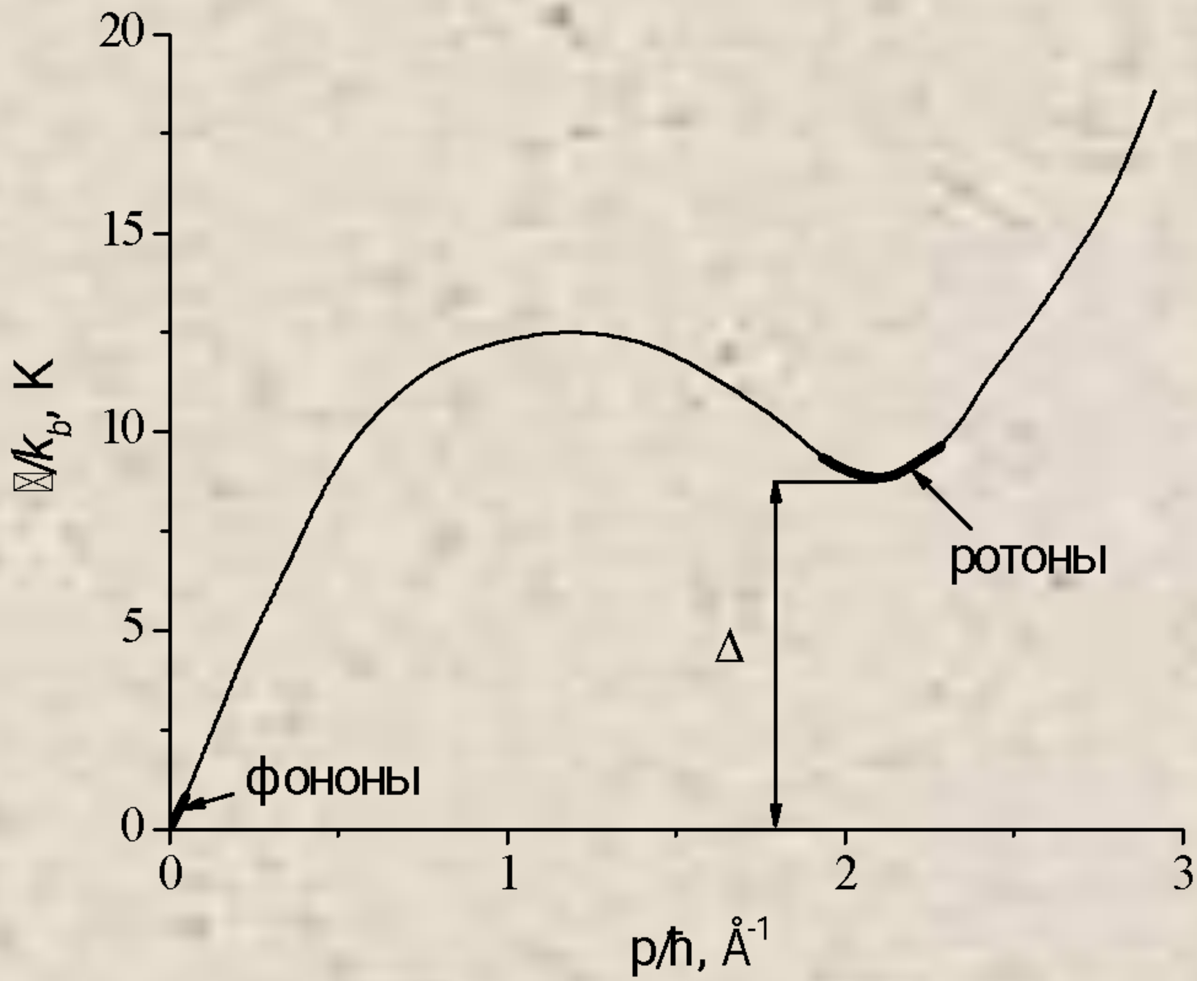
Экспериментальное определение вязкости



Плотности компонентов



Энергетический спектр



Уравнения двухскоростной гидродинамики

$$\rho' = \rho_n + \rho_s$$

$$\frac{\partial \rho'}{\partial t} + \operatorname{div}\left(\boldsymbol{j}\right) = 0 \qquad (4)$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0 \qquad (5)$$

$$\frac{\partial \boldsymbol{V}_s}{\partial t} + \operatorname{grad}\left(\varphi + \frac{\boldsymbol{V}_s^2}{2}\right) = 0 \qquad (6)$$

$$\frac{\partial\left(\rho' S\right)}{\partial t} + \operatorname{div}\left(\boldsymbol{F}\right) = 0 \qquad (7)$$

$$\frac{\partial\left(\rho' e_*\right)}{\partial t} + \operatorname{div}\left(\boldsymbol{E}\right) = 0 \qquad (8)$$

Требуется найти $\boldsymbol{j}, \Pi_{ik}, \varphi, \boldsymbol{E}, \boldsymbol{F}$

Преобразования уравнений двухскоростной гидродинамики

В покоящейся системе координат $\boldsymbol{j} = \rho'\boldsymbol{V}$;

в движущейся системе координат $\boldsymbol{j}_0 = \rho'\left(\boldsymbol{V} - \boldsymbol{V}_s\right)$.

$$\boldsymbol{j} = \boldsymbol{j}_0 + \rho'\boldsymbol{V}_s \qquad (9)$$

$$\rho' e_* = \rho'\left(e + \frac{\boldsymbol{V}^2}{2}\right) = \rho'\left[e + \frac{\left(\boldsymbol{V} - \boldsymbol{V}_s\right)^2}{2}\right] + \rho'\boldsymbol{V}\boldsymbol{V}_s - \frac{\rho'\boldsymbol{V}_s^2}{2} = \rho' e_{*0} + \rho'\boldsymbol{V}_s\left(\boldsymbol{V} - \frac{\boldsymbol{V}_s}{2}\right) =$$

$$= \rho' e_{*0} + \rho'\boldsymbol{V}_s\left(\boldsymbol{V} - \boldsymbol{V}_s\right) + \frac{\rho'\boldsymbol{V}_s^2}{2} = \rho' e_{*0} + \boldsymbol{j}_0\boldsymbol{V}_s + \frac{\rho'\boldsymbol{V}_s^2}{2} \qquad (10)$$

$$d\left(\rho e_{*0}\right) = \mu d\rho + Td\left(\rho S\right) + \left(\boldsymbol{V}_n - \boldsymbol{V}_s\right) d\boldsymbol{j}_0$$

Выражения для искомых параметров

$$\boldsymbol{j}_0 = \rho_n\left(\boldsymbol{V}_n - \boldsymbol{V}_s\right)$$

$$\boldsymbol{j} = \rho_n \boldsymbol{V}_n + \rho_s \boldsymbol{V}_s \qquad (12)$$

$$\Pi_{ik} = \delta_{ik} p + \rho_n V_{ni} V_{nk} + \rho_s V_{si} V_{sk} \qquad (13)$$

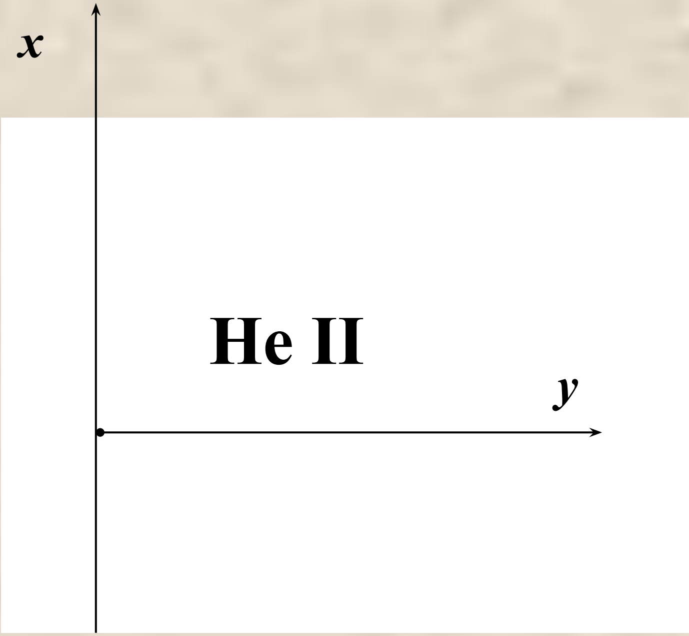
$$\varphi = \mu \qquad (14)$$

$$\boldsymbol{F} = \rho' S \boldsymbol{V}_n \qquad (15)$$

$$\boldsymbol{q} = T \cdot \boldsymbol{F}$$

$$\boldsymbol{q} = \rho' S T \boldsymbol{V}_n \qquad (16)$$

Граничные условия



$$\boldsymbol{q} = 0 \qquad (17a)$$

$$V_{ny}\Big|_{y=0} = 0 \qquad V_{sy}\Big|_{y=0} = 0$$

$$V_{nx}\Big|_{y=0} = 0$$

$$V_{ny}\Big|_{y=0} = \frac{q_y}{\rho' S T}$$

$$\boldsymbol{q} \neq 0 \qquad (17b)$$

$$V_{sy}\Big|_{y=0} = -\frac{\rho_n}{\rho_s}\frac{q_y}{\rho' S T}$$

Распространение звука в гелии II

$$\frac{\partial\rho}{\partial t}+\frac{\partial \boldsymbol{j}}{\partial x}=0 \quad (18)$$

$$\frac{\partial \boldsymbol{j}}{\partial t}+\frac{\partial p}{\partial x}=0 \quad (19)$$

$$\frac{\partial(\rho S)}{\partial t}+\rho S\frac{\partial V_n}{\partial x}=0 \quad (20)$$

$$\frac{\partial V_s}{\partial t}+\frac{\partial \mu}{\partial x}=0 \quad (21)$$

Дифференцируя (18) по t и (19) по x,

получаем $\frac{\partial^2\rho}{\partial t^2}=-\frac{\partial^2 \boldsymbol{j}}{\partial t\partial x}$ и $\frac{\partial^2 \boldsymbol{j}}{\partial x\partial t}=-\frac{\partial^2 p}{\partial x^2}$

$$\frac{\partial^2\rho}{\partial t^2}=\frac{\partial^2 p}{\partial x^2} \quad (22)$$

Из (20) $S\frac{\partial\rho}{\partial t}+\rho\frac{\partial S}{\partial t}+\rho S\frac{\partial V_n}{\partial x}=0$ (23) Из (18) и (23) $\frac{\partial\rho}{\partial t}=-\rho_n\frac{\partial V_n}{\partial x}-\rho_s\frac{\partial V_s}{\partial x}$ (24)

$$\frac{\partial\left(V_n-V_s\right)}{\partial x}=-\frac{\rho}{\rho_s S}\cdot\frac{\partial S}{\partial t} \quad (25)$$

$\frac{\partial\mu}{\partial x}=\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}-S\frac{\partial T}{\partial x}$ (26) Из (21) и (26) $\frac{\partial V_s}{\partial t}=-\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}+S\frac{\partial T}{\partial x}$ (27)

Из (19) и (27) $\frac{\partial p}{\partial x}=-\rho_n\frac{\partial V_n}{\partial t}-\rho_s\frac{\partial V_s}{\partial t}$ (28)

$$\frac{\partial\left(V_n-V_s\right)}{\partial t}=-\frac{\rho S}{\rho_n}\frac{\partial T}{\partial x} \quad (29)$$

Из (25) $\frac{\partial^2\left(V_n-V_s\right)}{\partial x\partial t}=-\frac{\rho}{\rho_s S}\cdot\frac{\partial^2 S}{\partial t^2}$ (30)

Из (29) $\frac{\partial^2\left(V_n-V_s\right)}{\partial x\partial t}=-\frac{\rho S}{\rho_n}\frac{\partial^2 T}{\partial x^2}$ (31)

$$\frac{\partial^2 S}{\partial t^2}=\frac{\rho_s S^2}{\rho_n}\frac{\partial^2 T}{\partial x^2} \quad (32)$$

$$\frac{\partial\rho}{\partial t}=\left(\frac{\partial\rho}{\partial p}\right)_T\frac{\partial p}{\partial t}+\left(\frac{\partial\rho}{\partial T}\right)_p\frac{\partial T}{\partial t}$$

$$\frac{\partial^2\rho}{\partial t^2}=\left(\frac{\partial\rho}{\partial p}\right)_T\frac{\partial^2 p}{\partial t^2}+\left(\frac{\partial\rho}{\partial T}\right)_p\frac{\partial^2 T}{\partial t^2}(33)$$

$$\frac{\partial^2\rho}{\partial x^2}=\left(\frac{\partial\rho}{\partial p}\right)_T\frac{\partial^2 p}{\partial x^2}+\left(\frac{\partial\rho}{\partial T}\right)_p\frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial^2 S}{\partial t^2}=\left(\frac{\partial S}{\partial p}\right)_T\frac{\partial^2 p}{\partial t^2}+\left(\frac{\partial S}{\partial T}\right)_p\frac{\partial^2 T}{\partial t^2}(34)$$

$$\frac{\partial^2 S}{\partial x^2}=\left(\frac{\partial S}{\partial p}\right)_T\frac{\partial^2 p}{\partial x^2}+\left(\frac{\partial S}{\partial T}\right)_p\frac{\partial^2 T}{\partial x^2}$$

$$\begin{cases}\left(\frac{\partial S}{\partial p}\right)_T\frac{\partial^2 p}{\partial t^2}+\left(\frac{\partial S}{\partial T}\right)_p\frac{\partial^2 T}{\partial t^2}=\frac{\rho_s S^2}{\rho_n}\frac{\partial^2 T}{\partial x^2}(32a)\\ \left(\frac{\partial\rho}{\partial T}\right)_p\frac{\partial^2 T}{\partial t^2}+\left(\frac{\partial\rho}{\partial p}\right)_T\frac{\partial^2 p}{\partial t^2}=\frac{\partial^2 p}{\partial x^2}(22a)\end{cases}$$

$$p = p_0 + \hat{p} \cdot \mathrm{Re}\left\{\exp\left[i\omega\left(t - \frac{x}{a}\right)\right]\right\} \quad T = T_0 + \hat{T} \cdot \mathrm{Re}\left\{\exp\left[i\omega\left(t - \frac{x}{a}\right)\right]\right\}$$

$$\frac{\partial^2 p}{\partial x^2} = \hat{p} \cdot \exp\left[i\omega\left(t - \frac{x}{a}\right)\right](i\omega)^2\left(-\frac{1}{a}\right)\left(-\frac{1}{a}\right) (35)$$

$$\frac{\partial^2 \rho}{\partial t^2} = \left\{\left(\frac{\partial \rho}{\partial p}\right)_T \hat{p} + \left(\frac{\partial \rho}{\partial T}\right)_p \hat{T}\right\} \cdot \exp\left[i\omega\left(t - \frac{x}{a}\right)\right](i\omega)^2 \ (36)$$

$$\left(\frac{\partial \rho}{\partial p}\right)_T \hat{p} + \left(\frac{\partial \rho}{\partial T}\right)_p \hat{T} = \frac{\hat{p}}{a^2} \text{ или } \quad \hat{p}\left[a^2\left(\frac{\partial \rho}{\partial p}\right)_T - 1\right] + a^2\left(\frac{\partial \rho}{\partial T}\right)_p \hat{T} = 0$$

$$\left(\frac{\partial S}{\partial p}\right)_T \hat{p} + \left(\frac{\partial S}{\partial T}\right)_p \hat{T} = \frac{\rho_s S^2}{\rho_n a^2} \cdot \hat{T} \text{ или } \quad \hat{p} \cdot a^2\left(\frac{\partial S}{\partial p}\right)_T + \left[a^2\left(\frac{\partial S}{\partial T}\right)_p - \frac{S^2 \rho_s}{\rho_n}\right] \cdot \hat{T} = 0$$

определитель $\left[a^2\left(\frac{\partial S}{\partial T}\right)_p - \frac{S^2 \rho_s}{\rho_n}\right]\left[a^2\left(\frac{\partial \rho}{\partial p}\right)_T - 1\right] - a^4\left(\frac{\partial \rho}{\partial T}\right)_p\left(\frac{\partial S}{\partial p}\right)_T = 0$

$$\left[a^2\left(\frac{\partial\rho}{\partial p}\right)_T-1\right]\left[a^2\left(\frac{\partial S}{\partial T}\right)_p-\frac{S^2\rho_s}{\rho_n}\right]=0$$

$$a_1=\sqrt{\left(\frac{\partial p}{\partial\rho}\right)_T}\ (37)$$

$$a_2=S\sqrt{\frac{\rho_s}{\rho_n\left(\partial S/\partial T\right)_p}}\quad(38)$$

$$\left(\frac{\partial p}{\partial\rho}\right)_S=\frac{c_p}{c_v}\left(\frac{\partial p}{\partial\rho}\right)_T$$

$$\left(\frac{\partial S}{\partial T}\right)_p=\frac{c_p}{T}$$

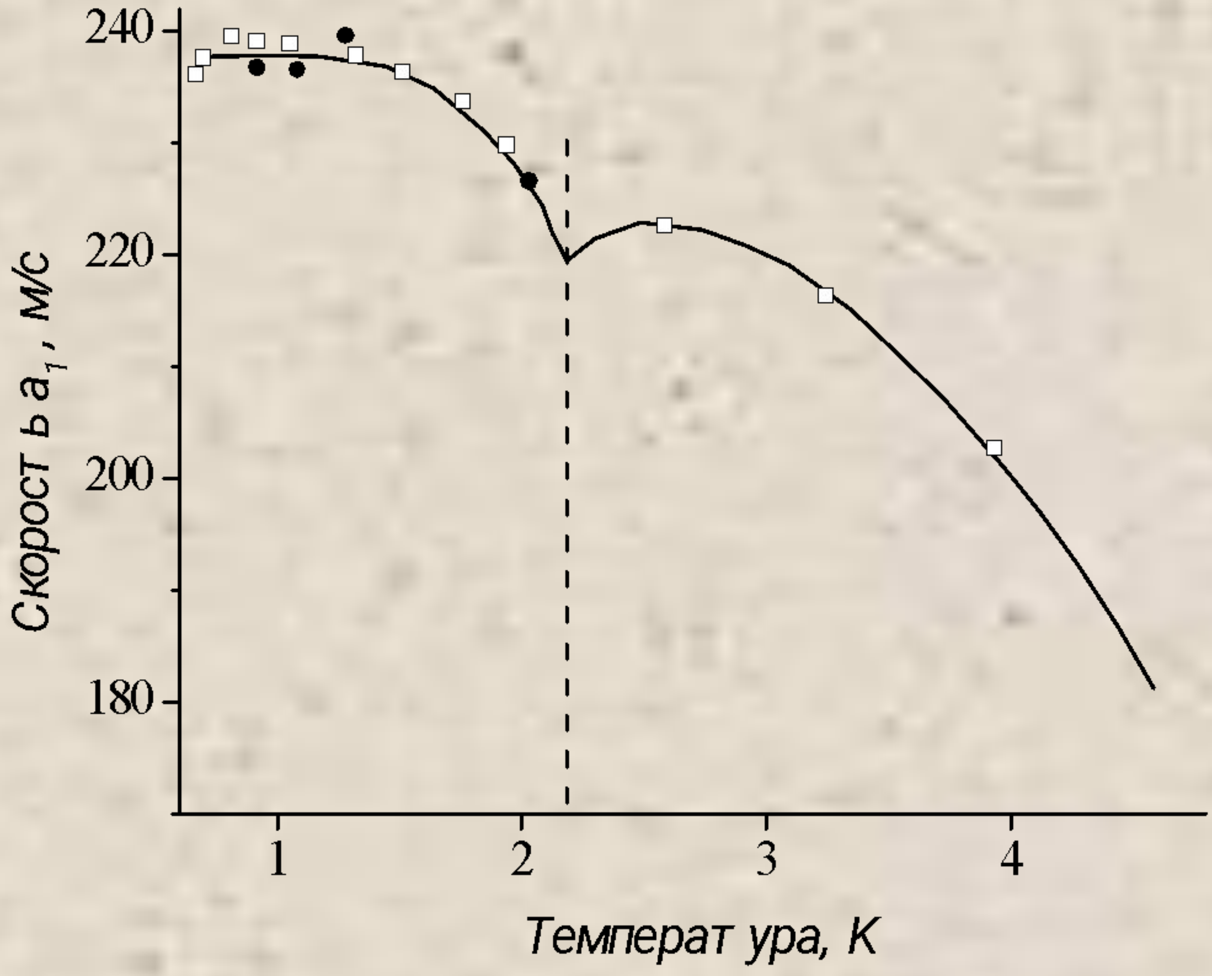
$$c_p=c_v+T\left(\frac{\partial p}{\partial T}\right)_V\left(\frac{\partial \upsilon}{\partial T}\right)_p$$

$$c_p=c_v=c$$

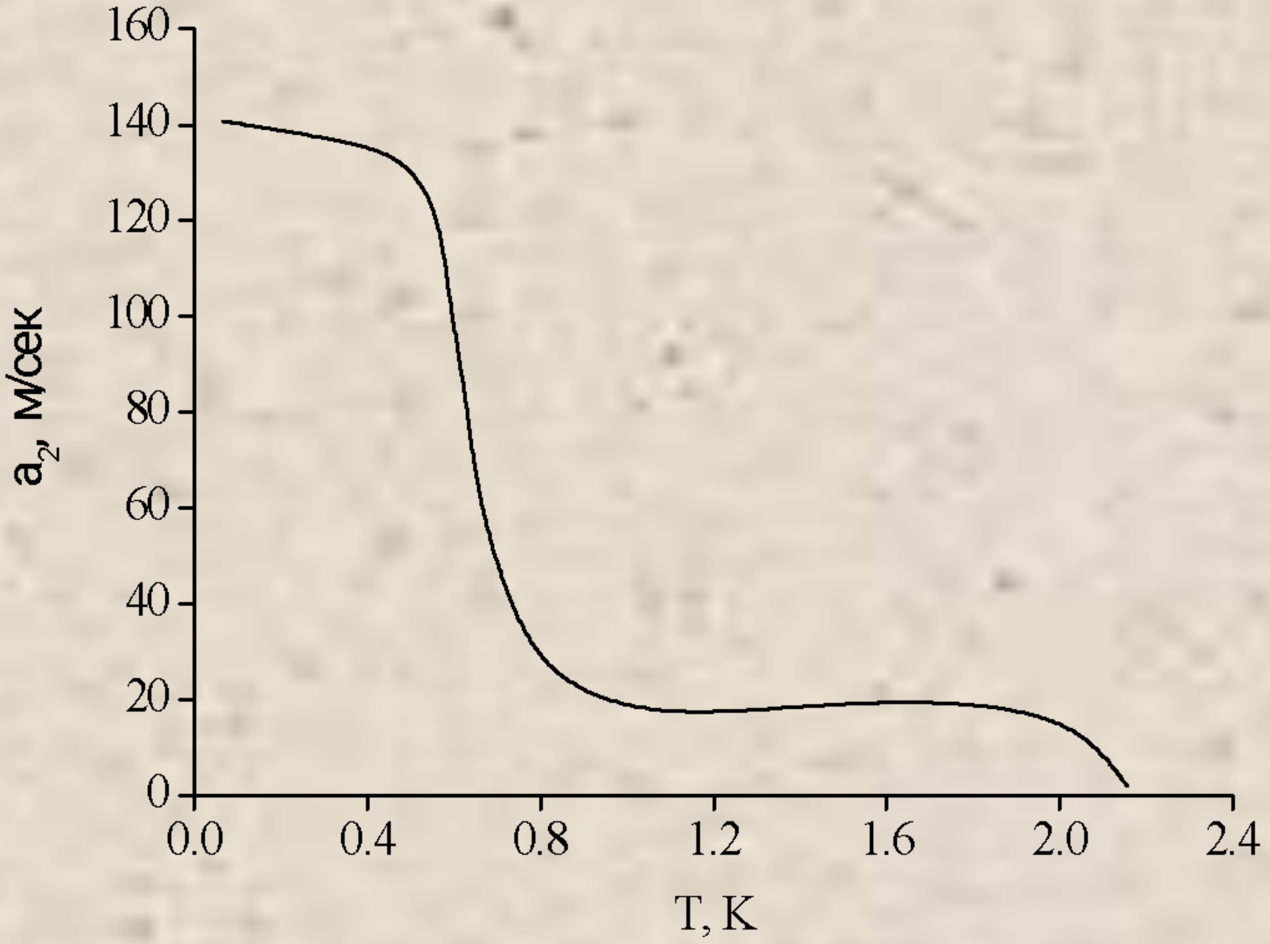
$$a_1=\sqrt{\left(\frac{\partial p}{\partial\rho}\right)_S}\ (37a)$$

$$a_2=S\sqrt{\frac{\rho_s T}{\rho_n c}}\quad(38a)$$

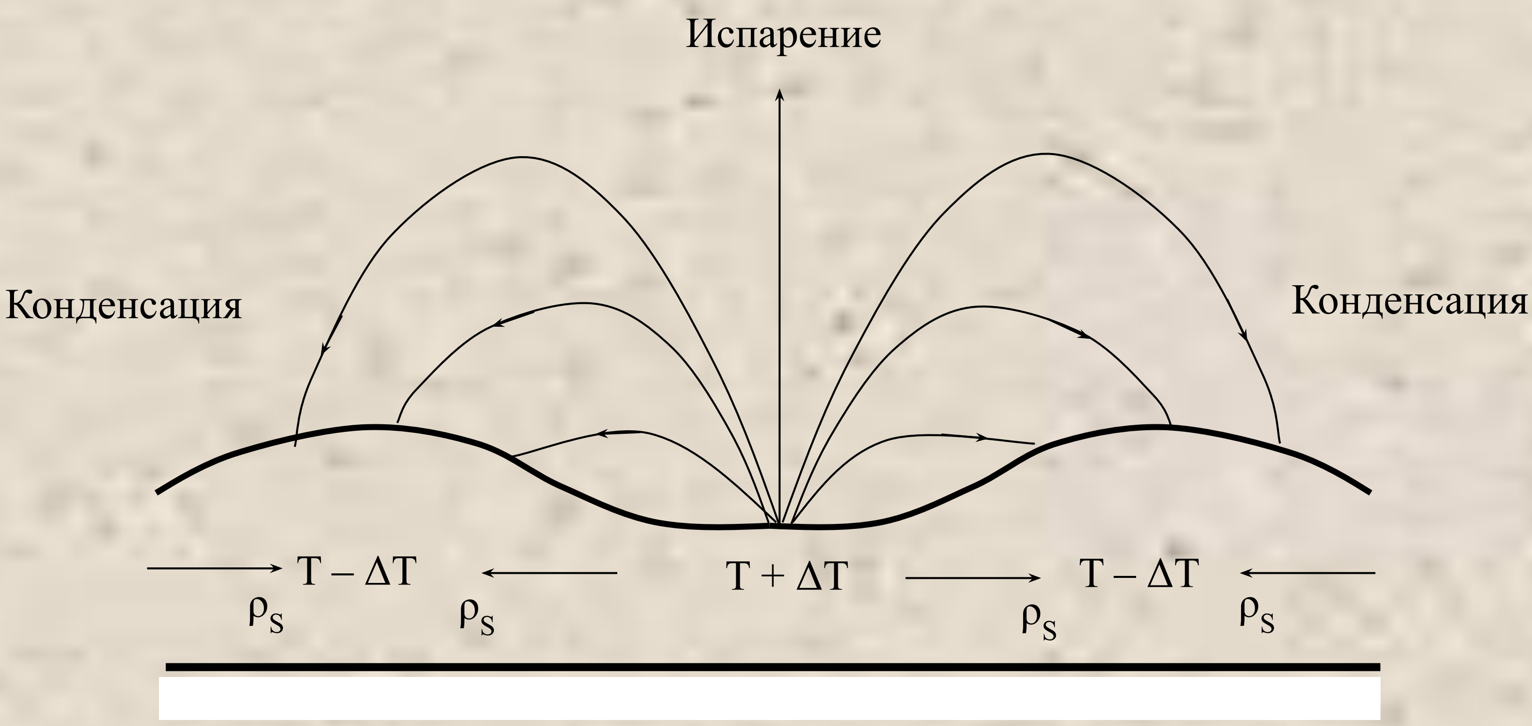
Скорость первого звука



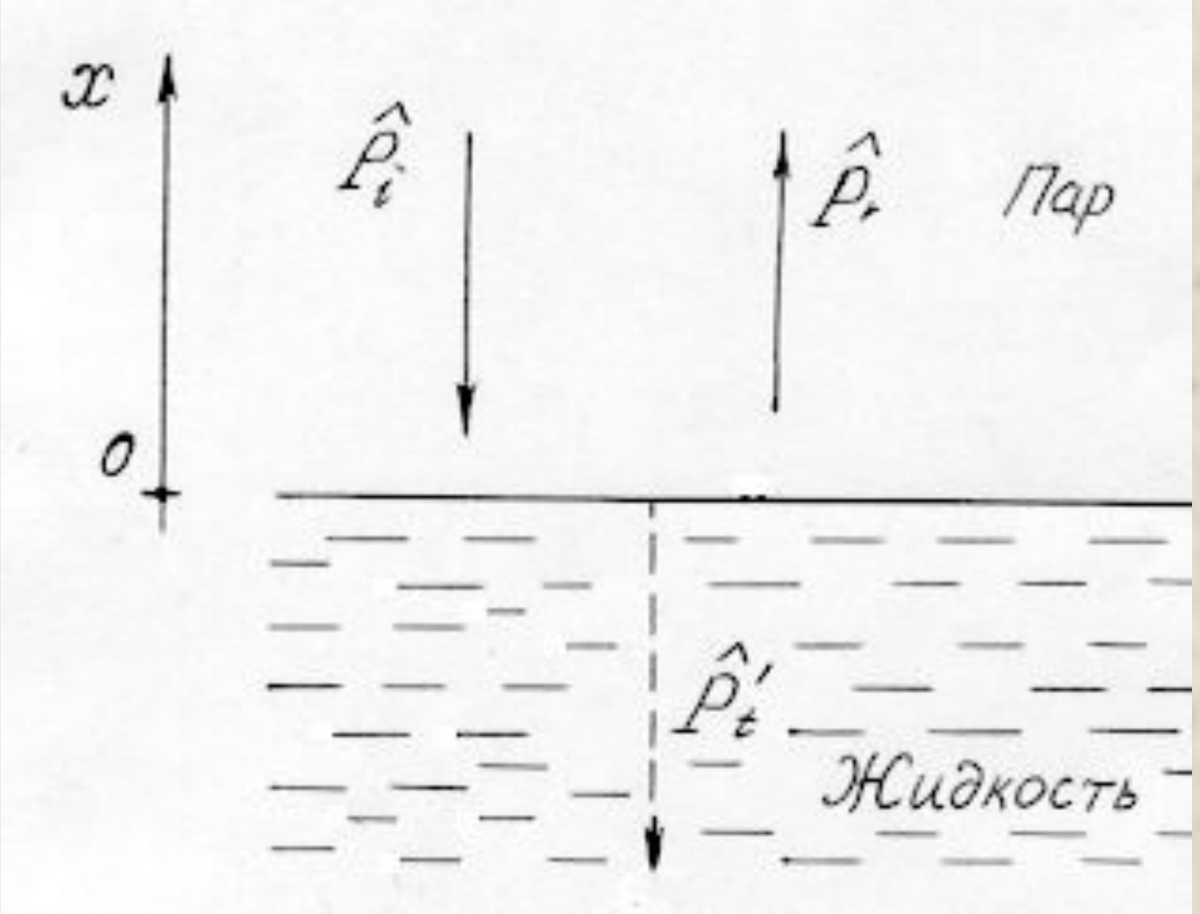
Скорость второго звука



Третий звук



Отражение звука от межфазной поверхности He II-пар



Коэффициент отражения звука

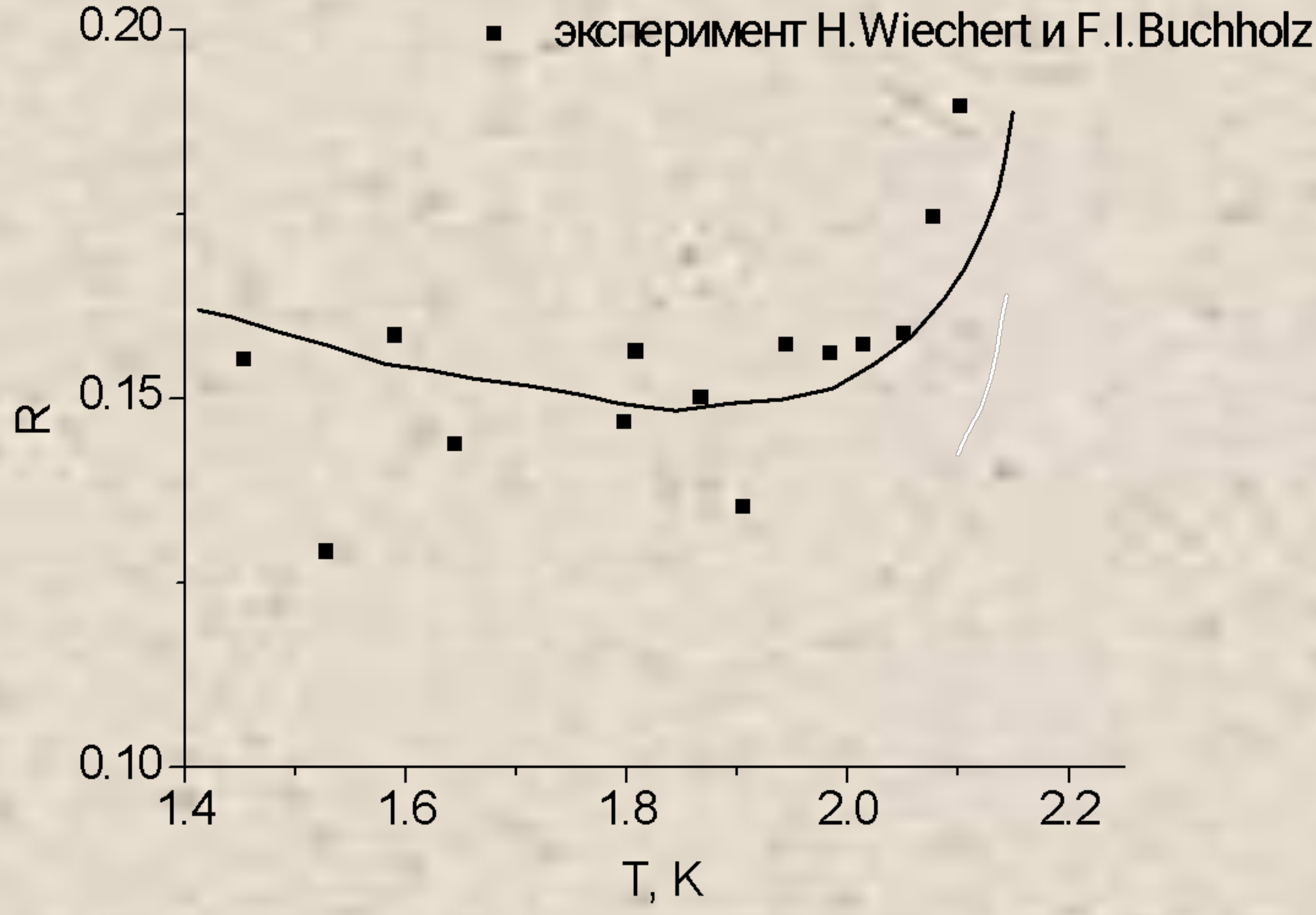
$$R = \frac{1 - \chi - 0{,}86 \cdot {(\rho' - \rho)}\big/{(1 + A)\rho'}}{1 + \chi + 0{,}86 \cdot {(\rho' - \rho)}\big/{(1 + A)\rho'}}$$

$$\chi = {\rho a}\big/{\rho' a'}$$

$$A = \frac{0{,}86 \cdot r}{\rho' a_2 C a} \cdot \left(\frac{\partial p}{\partial T}\right)_{нас}$$

$$\left(\frac{\partial p}{\partial T}\right)_{нас} = \frac{r \cdot \rho' \rho}{T_0 (\rho' - \rho)}$$

Коэффициент отражения звука



Описание теплопереноса в He II

$$\frac{\partial(\rho' S)}{\partial t} + \operatorname{div}(\rho' S \boldsymbol{V}_n) = 0 \quad (51) \qquad \frac{\partial(\rho' S)}{\partial t} = 0$$

$$\operatorname{div}\boldsymbol{V}_n = 0 \quad (52)$$

$$\boldsymbol{j} = \rho_n \boldsymbol{V}_n + \rho_s \boldsymbol{V}_s \qquad \rho_n = \boldsymbol{const} \quad \rho_s = \boldsymbol{const}$$

$$\operatorname{div}\boldsymbol{V}_s = 0 \quad (53)$$

$$\Pi_{lk} = \delta_{lk} p + \rho_n V_{nl} V_{nk} + \rho_s V_{sl} V_{sk}$$

$$\frac{\partial(\rho_n V_{nl})}{\partial t} + \frac{\partial(\rho_s V_{sl})}{\partial t} + \frac{\partial p}{\partial x_l} + \frac{\partial(\rho_n V_{nl} V_{nk})}{\partial x_k} + \frac{\partial(\rho_s V_{sl} V_{sk})}{\partial x_k} = 0 \quad (54)$$

$$\rho_n \frac{D\boldsymbol{V}_n}{dt} + \rho_s \frac{D\boldsymbol{V}_s}{dt} + \nabla p = 0 \quad (55)$$

$$\frac{\partial \boldsymbol{V}_s}{\partial t} + \nabla\left(\mu + \frac{\boldsymbol{V}_s^2}{2}\right) = 0 \quad (56) \qquad (\boldsymbol{V}_s \cdot \nabla)\boldsymbol{V}_s = \nabla\left(\frac{\boldsymbol{V}_s^2}{2}\right) + \operatorname{rot}\boldsymbol{V}_s \times \boldsymbol{V}_s \qquad \operatorname{rot}\boldsymbol{V}_s = 0$$

$$\frac{\partial \boldsymbol{V}_s}{\partial t} + (\boldsymbol{V}_s \cdot \nabla)\boldsymbol{V}_s + \nabla\mu = 0 \quad (57) \qquad \nabla\mu = \frac{\nabla p}{\rho} - S\nabla T$$

Преобразованная система уравнений

$$\rho_s \frac{D\mathbf{V_s}}{dt} = -\frac{\rho_s}{\rho}\nabla p + \rho_s S \nabla T \quad (58)$$

$$\rho_n \frac{D\mathbf{V_n}}{dt} = -\frac{\rho_n}{\rho}\nabla p - \rho_s S \nabla T \quad (59)$$

$$\begin{cases} \rho_s \dfrac{D\mathbf{V_s}}{dt} = -\dfrac{\rho_s}{\rho}\nabla p + \rho_s S \nabla T - \mathbf{F}_{вз.тр} & (58\ \ ) \\ \rho_n \dfrac{D\mathbf{V_n}}{dt} = -\dfrac{\rho_n}{\rho}\nabla p - \rho_s S \nabla T + \eta \Delta \mathbf{V} + \mathbf{F}_{вз.тр} & (59\ \ ) \end{cases}$$

Стационарный теплоперенос в He II при безвихревом сверхтекучем движении

$$\begin{cases} -\dfrac{\rho_s}{\rho}\nabla p + \rho_s S \nabla T = 0 & (60) \\ -\dfrac{\rho_n}{\rho}\nabla p - \rho_s S \nabla T + \eta \Delta \boldsymbol{V_n} = 0 & (61) \end{cases}$$

$$\nabla p = \eta \Delta \boldsymbol{V_n} \quad (62)$$

При ламинарном нормальном движении

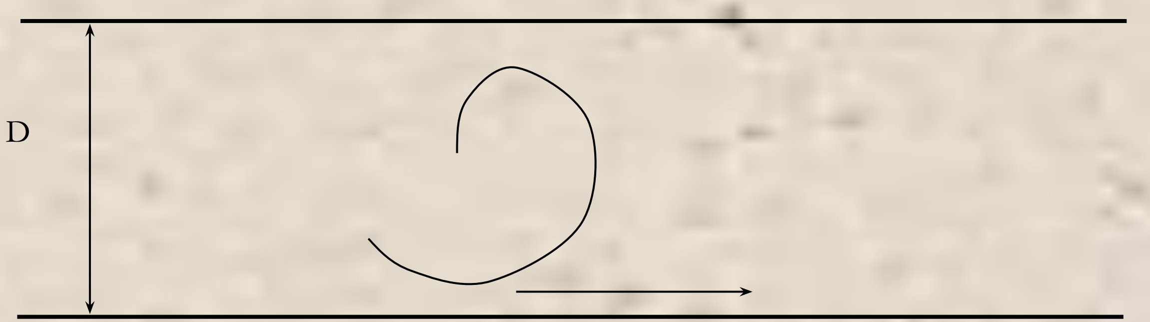
$$\bar{V}_n = \frac{(p_1 - p_2)R^2}{8\eta l}$$

$$\nabla p = \rho S \nabla T$$

$$q = \rho S T \bar{V}_n = \rho S T \frac{\rho S (T_1 - T_2) R^2}{8\eta l} = -(\rho S)^2 T \frac{(T_2 - T_1) R^2}{8\eta l} = -\frac{(\rho S)^2 T R^2}{8\eta} \nabla T \quad (63)$$

$$C = \frac{8\eta}{(\rho S)^2 T R^2}$$

Вихри сверхтекучего компонента Критическая скорость сверхтекучего движения



$$\Gamma = \pi d V_s$$

$$\Gamma = n\frac{h}{m}$$

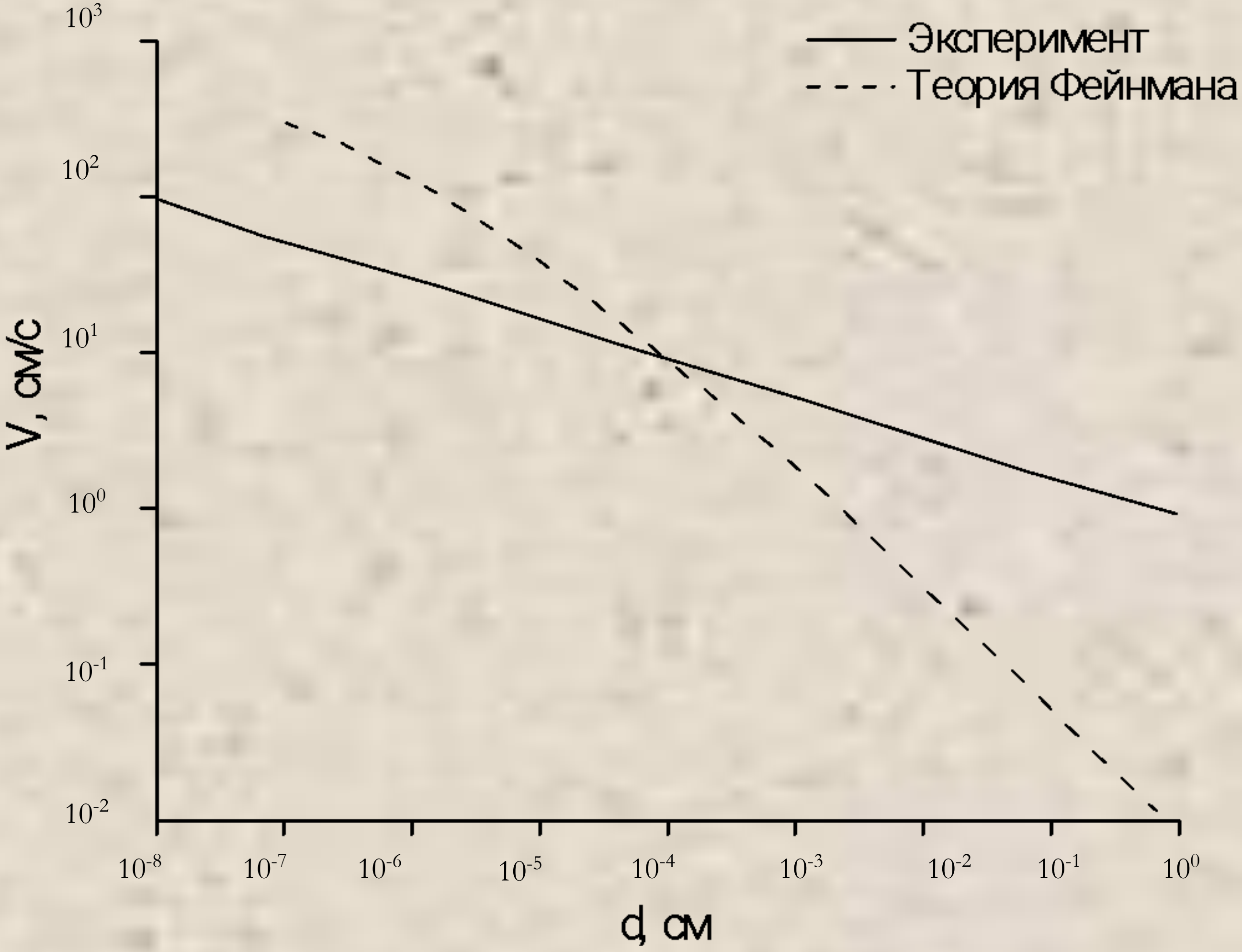
$$\pi d V_s = n\frac{h}{m}$$

$$d = D \qquad n = 1$$

$$V_{scr} = \frac{h}{\pi D m} \quad (65)$$

$$V_{scr} = \frac{h}{\pi D m}\left(\ln\frac{4D}{a_0} - K'\right) \quad (66)$$

$$V_{scr} = \frac{1}{D^{1/4}} \quad (67)$$



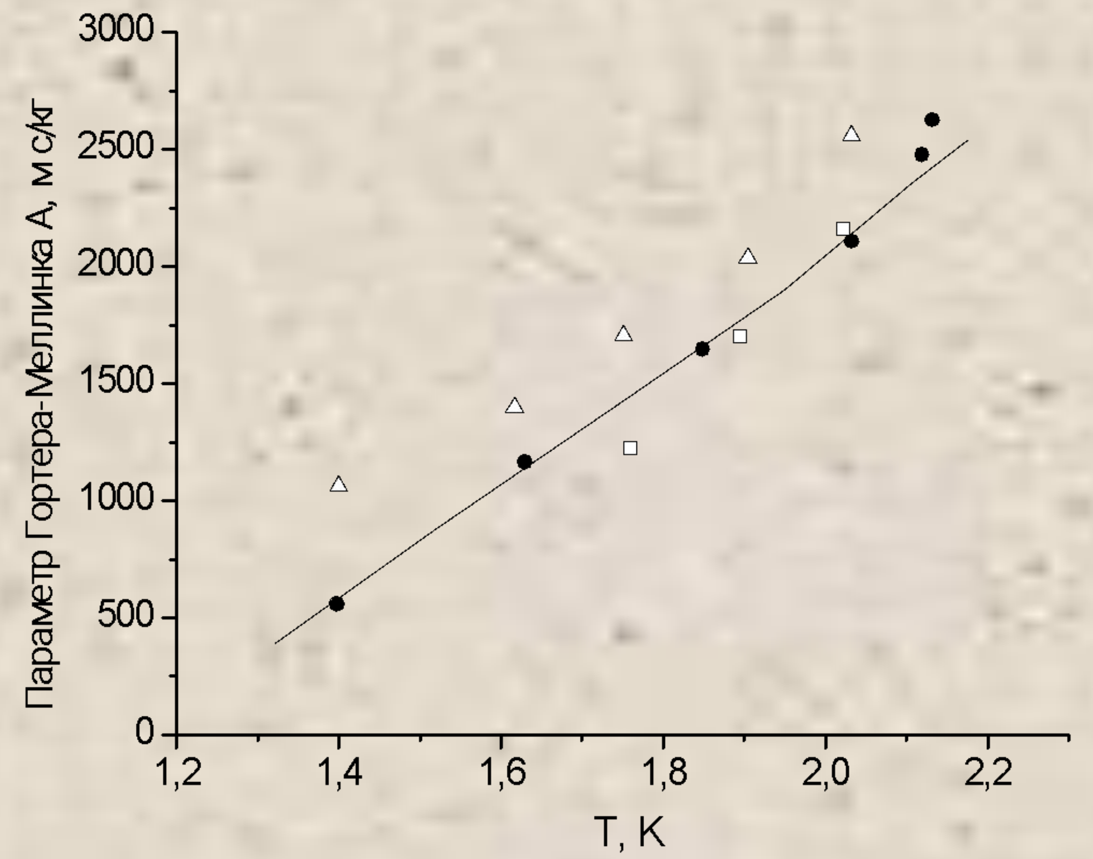
Эксперимент
Теория Фейнмана
V, см/с
d, см
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0

Сила взаимного трения

$$f \sim -\frac{\rho_n \rho_s}{\rho^2}\eta\left(\boldsymbol{V}_n - \boldsymbol{V}_s\right)$$

$$L \sim \left(\frac{\rho_n}{\rho}\right)^2 \left(\boldsymbol{V}_n - \boldsymbol{V}_s\right)^2$$

$$\boldsymbol{F}_{вз.тр} = f \cdot L$$



$$\boldsymbol{F}_{вз.тр} = -A\rho_n\rho_s\left(\boldsymbol{V}_n - \boldsymbol{V}_s\right)^3 \quad (68)$$

Расчет стационарного теплопереноса в He II с учетом силы взаимного трения

$$\begin{cases} -\dfrac{\rho_s}{\rho}\nabla p + \rho_s S \nabla T + A\rho_n\rho_s\left(\boldsymbol{V}_n - \boldsymbol{V}_s\right)^3 = 0 \quad (69) \\ -\dfrac{\rho_n}{\rho}\nabla p - \rho_s S \nabla T + \eta\Delta\boldsymbol{V}_n - A\rho_n\rho_s\left(\boldsymbol{V}_n - \boldsymbol{V}_s\right)^3 = 0 \quad (70) \end{cases}$$

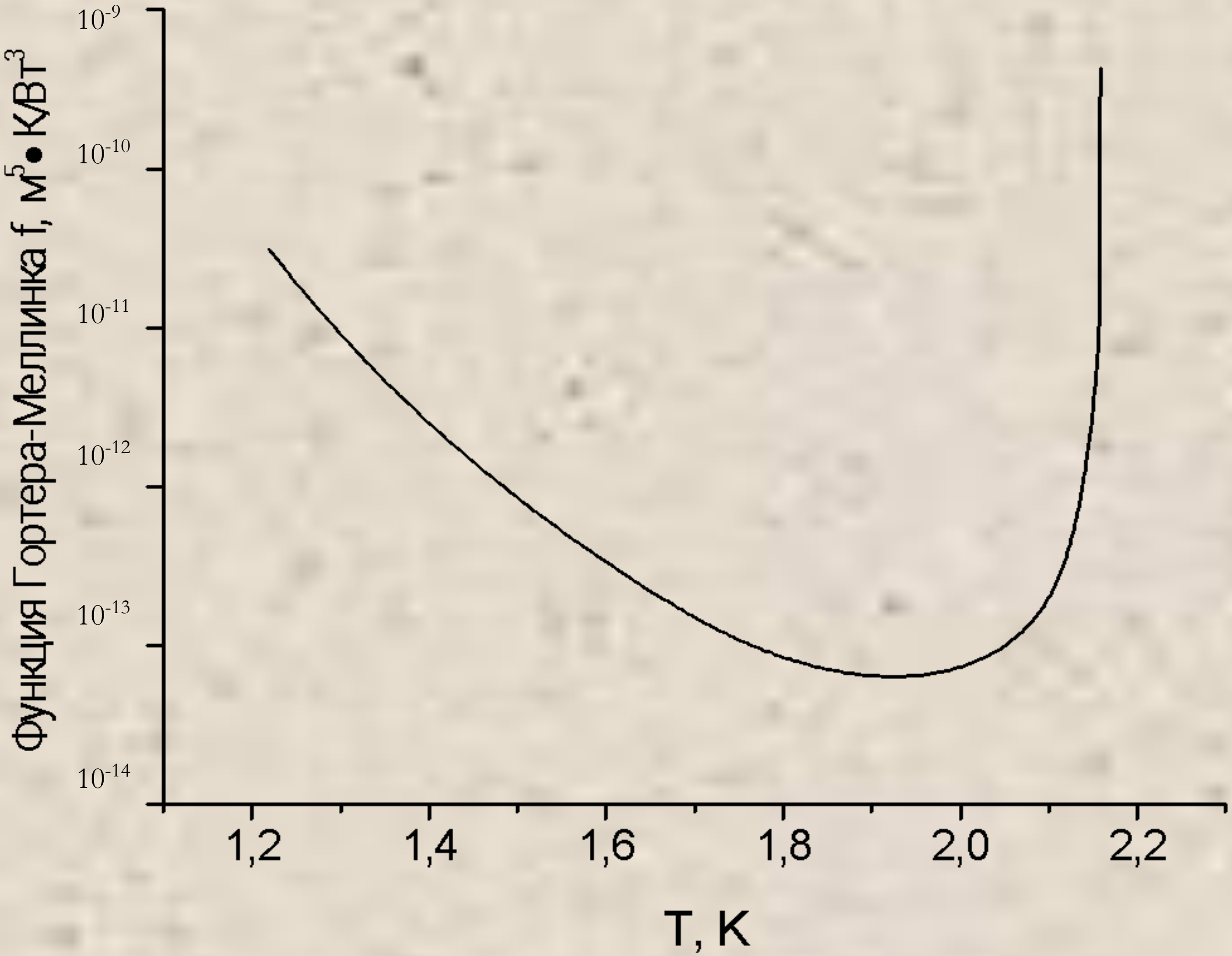
$$\nabla T = -\frac{A\rho_n\left(\boldsymbol{V}_n - \boldsymbol{V}_s\right)^3}{S}$$

$$\nabla T = -\frac{A\rho_n\rho^3\boldsymbol{V}_n^3}{S\rho_s^3}$$

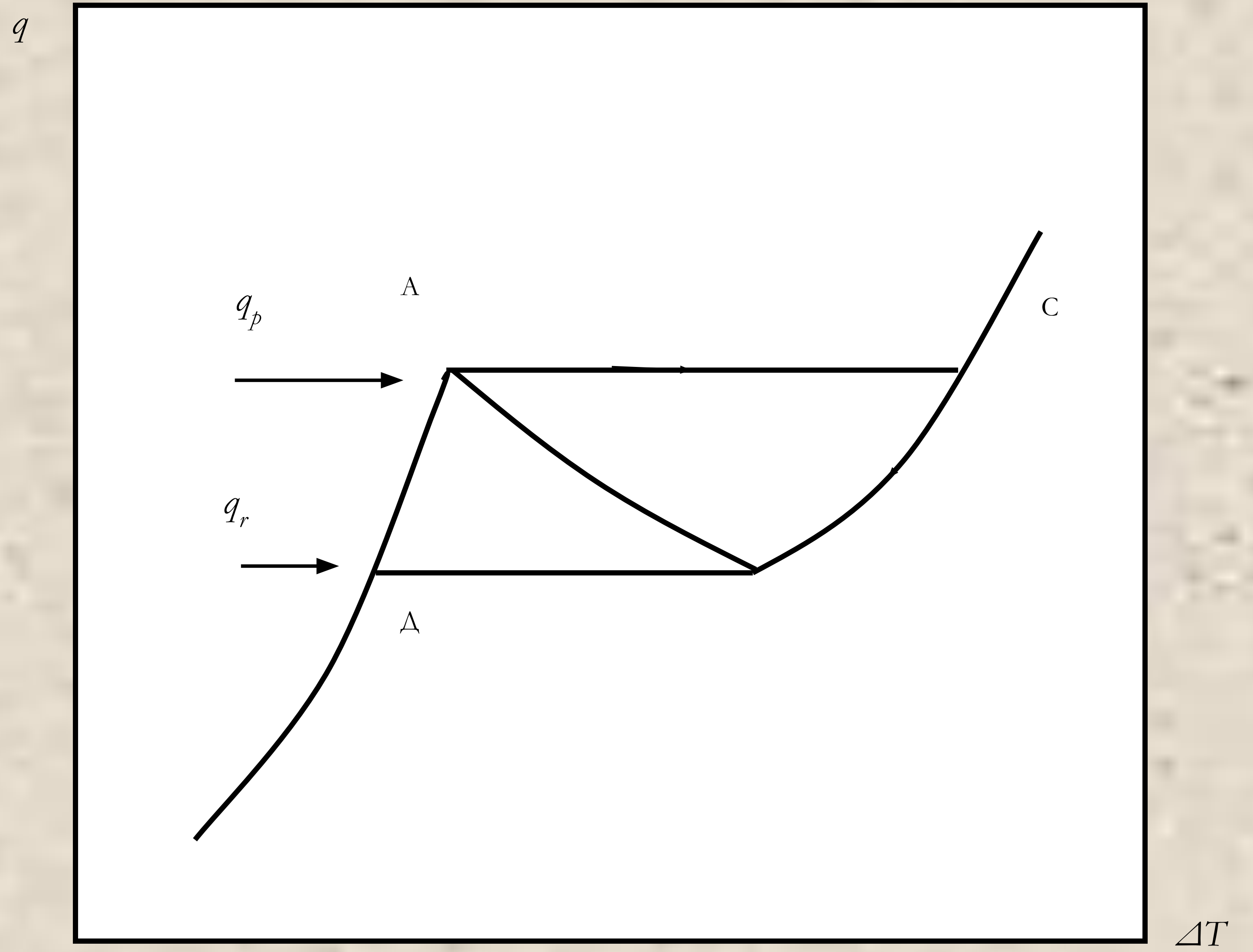
$$\nabla T = -\frac{A\rho_n}{\rho_s^3 S^4 T^3}\boldsymbol{q}^3 = -f\left(T\right)\cdot\boldsymbol{q}^3 \quad (71)$$

$$\nabla T = -C\boldsymbol{q} - f\left(T\right)\cdot\boldsymbol{q}^3 \quad (71a)$$

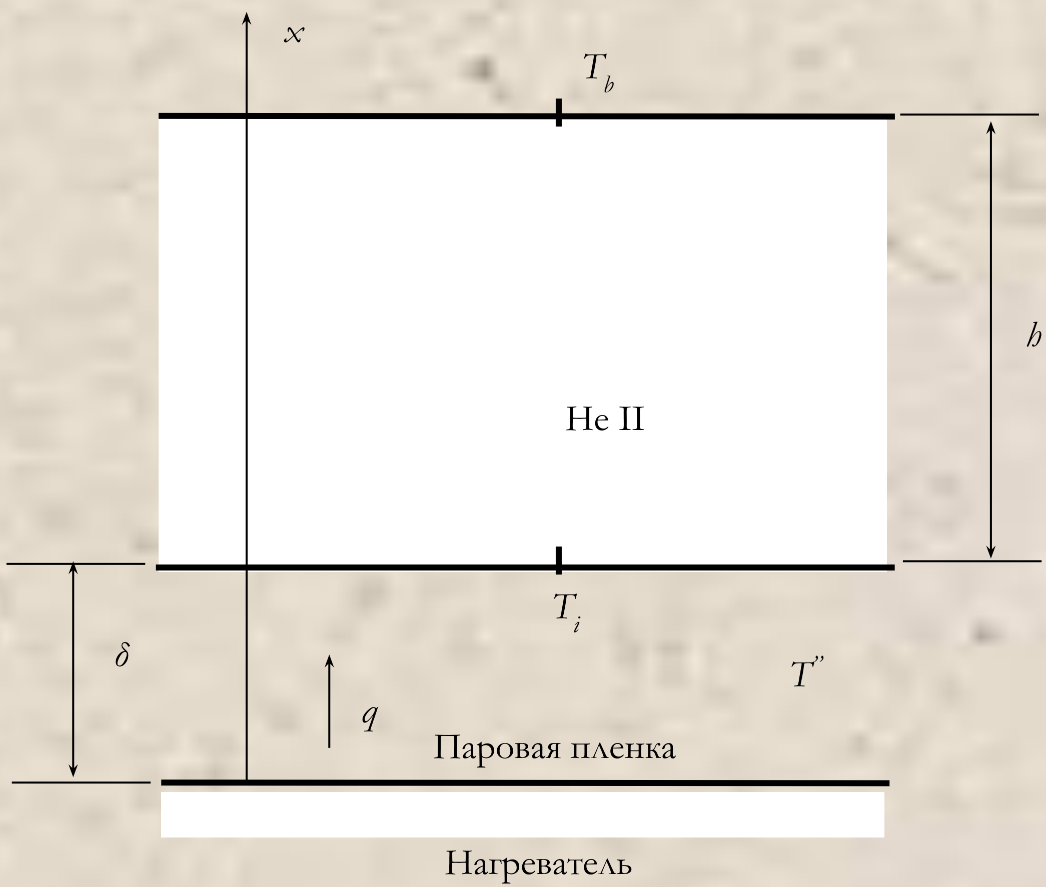
Функция Гортера-Меллинка



Кривая кипения He II



Кипение гелия II



$$\boldsymbol{q_R} = 2,27 \cdot \left(\boldsymbol{p''} - \boldsymbol{p}_{\text{нас}}\left(\boldsymbol{T_i}\right)\right) \cdot \sqrt{2\boldsymbol{R}\boldsymbol{T_i}} \quad (72)$$

$$\tilde{q}_R = {q_R}\Big/\left(p_{\text{Нас}}\left(T_i\right) \cdot \sqrt{2RT_i}\ \right) << 1$$

$$\boldsymbol{q_R} = 8 \cdot \left(\frac{\boldsymbol{p''}}{\boldsymbol{p}_{\text{нас}}\left(\boldsymbol{T_i}\right)} - 1 \right) \cdot \boldsymbol{p''} \sqrt{\frac{\boldsymbol{R}\boldsymbol{T_i}}{2\pi}} \quad (72\boldsymbol{a})$$

$$\frac{\boldsymbol{dT}}{\boldsymbol{dr}} = -\boldsymbol{f}\left(\boldsymbol{T}\right) \cdot \boldsymbol{q}^3 \quad (73)$$

$$\boldsymbol{q_i \cdot r_i = q \cdot r} \quad (74)$$

$$\boldsymbol{q = \frac{q_R \cdot r_w}{r}} \quad (74\boldsymbol{a})$$

$$\boldsymbol{\frac{dT}{dr} = -f(T) \cdot \frac{{q_R}^3 \cdot {r_w}^3}{r^3}} \quad (73\boldsymbol{a})$$

$$\int_{T_i}^{T_\infty} \frac{dT}{f(T)} = -{q_R}^3 \cdot {r_w}^3 \cdot \int_{r_w}^{\infty} \frac{dr}{r^3}$$

$$\int_{T_i}^{T_b} \frac{dT}{f(T)} = -\int_{T_b}^{T_i} \frac{dT}{f(T)} = -\frac{T_i - T_b}{\tilde{f}(T)}$$

$$\tilde{f}(T) = \frac{T_i - T_b}{\int_{T_b}^{T_i} \frac{dT}{f(T)}}$$

$$-\frac{T_i - T_b}{\tilde{f}(T)} = \frac{q_R^{\,3} \cdot r_w^{\,3}}{2r^2}\Big|_{r_w}^{\infty} = -\frac{q_R^{\,3} \cdot r_w^{\,3}}{2r_w^{\,2}} = -\frac{q_R^{\,3} \cdot r_w}{2}$$

$$T_i - T_b = f(T) \cdot \frac{q_R^{\,3} \cdot r_w}{2} \quad (73\ \ )$$

$$p'' = p_b + \rho g h \quad (75)$$

$$\Delta T = \left({dT}/{dp} \right)_{\text{нас.}} \cdot \rho g h$$

Определение восстановительного потока