Rank of a matrix. Theorem of Kronecker-Capelli

Consider a system of n linear equations with n variables $(x_1, x_2, \ldots, x_n)$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases} \quad (*)$$

Consider

$$A(n;n) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \qquad X(n;1) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad B(n;1) = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Then the system (*) is written in a matrix representation: $A(n; n) \cdot X(n; 1) = B(n; 1)$

$X = A^{-1} \cdot B$.

A matrix $A(n; n)$ is called *regular* if its determinant is not equal to zero, i.e.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0.$$

A matrix $A^{-1}(n; n)$ is called *inverse* to a matrix $A(n; n)$ if the product

$$A(n; n) \cdot A^{-1}(n; n) = A^{-1}(n; n) \cdot A(n; n) = E(n; n),$$

$$A^{-1}=\frac{1}{\Delta}\cdot\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}.$$

Example . Find the inverse matrix to the matrix

$$A=\begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix}. \qquad 1)\ \Delta=\begin{vmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{vmatrix}=12-2+0-0-8+3=5\neq 0.$$

2) $A_{11}=5$; $A_{12}=10$; $A_{13}=0$; $A_{21}=4$; $A_{22}=12$; $A_{23}=1$; $A_{31}=-1$; $A_{32}=-3$; $A_{33}=1$.

$$3)\ A^{*}=\begin{pmatrix} 5 & 10 & 0 \\ 4 & 12 & 1 \\ -1 & -3 & 1 \end{pmatrix}.\ 4)\ A^{*T}=\begin{pmatrix} 5 & 4 & -1 \\ 10 & 12 & -3 \\ 0 & 1 & 1 \end{pmatrix}.\ 5)\ A^{-1}=\frac{1}{5}\cdot\begin{pmatrix} 5 & 4 & -1 \\ 10 & 12 & -3 \\ 0 & 1 & 1 \end{pmatrix}.$$

$$A\cdot A^{-1}=\frac{1}{5}\cdot\begin{pmatrix}3 & -1 & 0\\ -2 & 1 & 1\\ 2 & -1 & 4\end{pmatrix}\cdot\begin{pmatrix}5 & 4 & -1\\ 10 & 12 & -3\\ 0 & 1 & 1\end{pmatrix}=\frac{1}{5}\cdot\begin{pmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Rank of a matrix.

Consider a matrix of the dimension $m\times n$: $A(m;n)=\begin{pmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \dots & \dots & \dots & \dots\\ a_{m1} & a_{m2} & \dots & a_{mn}\end{pmatrix}$

The rank of a matrix A is the greatest order of its non-equal to zero minors. The rank of a matrix is denoted by *Rank A* or *r(A)*.

Theorem. If there is a non-equal to zero minor of the *r*-th order in a matrix *A* and all its bordering minors of the *r*+1-th order are equal to zero then the rank of *A* is equal to *r*, i.e. *r(A)=r*.

Theorem. The rank of a matrix doesn't change if:

a) All the rows are replaced by the corresponding columns and vice versa;

b) Replace two arbitrary rows (columns);

c) Multiply (divide) each element of a row (column) on the same non-zero number;

d) Add to (subtract from) elements of a row (column) the corresponding elements of any other row (column) multiplied on the same non-zero number.

Theorem of Kronecker-Capelli. A system of linear equations is consistent if the rank of the basic matrix A equals the rank of the extended matrix C, i.e. $Rank\ A = Rank\ C$. Moreover:

- 1) If $Rank\ A = Rank\ C = n$ (where n is the number of variables in the system) then the system has a unique solution.
- 2) If $Rank\ A = Rank\ C < n$ then the system has infinitely many solutions.

Solving a system of linear equations by the Gauss method

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z + a_{14}u = a_{15} & (a) \\ a_{21}x + a_{22}y + a_{23}z + a_{24}u = a_{25} & (b) \\ a_{31}x + a_{32}y + a_{33}z + a_{34}u = a_{35} & (c) \\ a_{41}x + a_{42}y + a_{43}z + a_{44}u = a_{45} & (d) \end{cases}$$

Suppose that $a_{11} \neq 0$ (if $a_{11} = 0$ then we change the order of equations by choosing as the first equation such an equation in which the coefficient of x is not equal to zero).

I step: divide the equation *(a)* on a_{11}, then multiply the obtained equation on a_{21} and subtract from *(b)*; further multiply $(a)/a_{11}$ on a_{31} and subtract from *(c)*; at last, miltiply $(a)/a_{11}$ on a_{41} and subtract from *(d)*.

$$\begin{cases} x + b_{12}y + b_{13}z + b_{14}u = b_{15} & (e) \\ b_{22}y + b_{23}z + b_{24}u = b_{25} & (f) \\ b_{32}y + b_{33}z + b_{34}u = b_{35} & (g) \\ b_{42}y + b_{43}z + b_{44}u = b_{45} & (i) \end{cases}$$

where b_{ij} are obtained from a_{ij} by the following formulas:

$b_{1j} = a_{1j} / a_{11}$ $(j = 2, 3, 4, 5)$;

$b_{ij} = a_{ij} - a_{i1} \cdot b_{1j}$ $(i = 2, 3, 4; j = 2, 3, 4, 5)$.

II step: do the same actions with *(f)*, *(g)*, *(i)* (as with *(a)*, *(b)*, *(c)*, *(d)*) and etc.

As a final result the initial system will be transformed to a so-called step form:

$$\begin{cases} x + b_{12}y + b_{13}z + b_{14}u = b_{15} \\ y + c_{23}z + c_{24}u = c_{25} \\ z + d_{34}u = d_{35} \\ u = e_{45} \end{cases}$$

Example 1.

$$\begin{cases} 3x + 2y + z = 5 \\ x + y - z = 0 \\ 4x - y + 5z = 3 \end{cases}$$

Interchange the first and the second equations of the system:

$$\begin{cases} x + y - z = 0 \\ 3x + 2y + z = 5 \\ 4x - y + 5z = 3 \end{cases}$$

Subtract from the second equation the first equation multiplied on 3; also subtract from the third equation the first equation multiplied on 4. We obtain:

$$\begin{cases} x + y - z = 0 \\ -y + 4z = 5 \\ -5y + 9z = 3 \end{cases}$$

Further subtract from the third equation the second equation multiplied on 5:

$$\begin{cases} x + y - z = 0 \\ -y + 4z = 5 \\ -11z = -22 \end{cases}$$

Multiply the second equation on (-2), and the third – divide on (-11):

$$\begin{cases} x + y - z = 0 \\ y - 4z = -5 \\ z = 2 \end{cases}$$

The system of equations has a triangular form, and consequently it has a unique decision. From the last equation we have $z = 2$; substituting this value in the second equation, we receive $y = 3$ and, at last from the first equation we find $x = -1$.

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