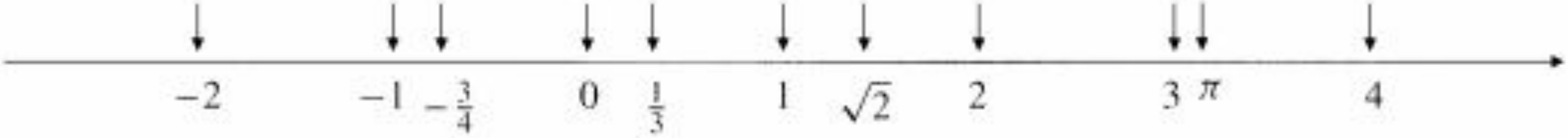
Real numbers and real line

Calculus depends on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, for example,

$$
\begin{aligned}
5 &= 5.00000\ldots \\
-\tfrac{3}{4} &= -0.750000\ldots \\
\tfrac{1}{3} &= 0.3333\ldots \\
\sqrt{2} &= 1.4142\ldots \\
\pi &= 3.14159\ldots
\end{aligned}
$$

The real numbers can be represented geometrically as points on a number line, which we call the **real line**, shown in Figure P.1. The symbol $\mathbb{R}$ is used to denote either the real number system or, equivalently, the real line.



The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. You are already familiar with the *algebraic properties*; roughly speaking, they assert that real numbers can be added, subtracted, multiplied, and divided (except by zero) to produce more real numbers and that the usual rules of arithmetic are valid.

The *order properties* of the real numbers refer to the order in which the numbers appear on the real line. If x lies to the left of y, then we say that "x is less than y" or "y is greater than x." These statements are written symbolically as $x < y$ and $y > x$, respectively. The inequality $x \leq y$ means that either $x < y$ or $x = y$. The order properties of the real numbers are summarized in the following *rules for inequalities*:

The set of real numbers has some important special subsets:

(i) the **natural numbers** or **positive integers**, namely, the numbers $1, 2, 3, 4, \ldots$

(ii) the **integers**, namely, the numbers $0, \pm 1, \pm 2, \pm 3, \ldots$

(iii) the **rational numbers**, that is, numbers that can be expressed in the form of a fraction m/n, where m and n are integers, and $n \neq 0$.

The rational numbers are precisely those real numbers with decimal expansions that are either:

(a) terminating, that is, ending with an infinite string of zeros, for example, $3/4 = 0.750000\ldots$, or

(b) repeating, that is, ending with a string of digits that repeats over and over, for example, $23/11 = 2.090909\ldots = 2.\overline{09}$. (The bar indicates the pattern of repeating digits.)

Real numbers that are not rational are called *irrational numbers*.

For example, $\frac{1}{3}$ is a repeating decimal.

$$
\begin{array}{r|l}
1.000 & 3 \\
-\ \ 9 & \\
\hline
10 & \\
-\ \ 9 & \\
\hline
10 & \\
-\ \ 9 & \\
\hline
10 &
\end{array}
$$

Thus, $\frac{1}{3} = 0.333\ldots$.

we can write $\frac{1}{3} = 0.3333\ldots = 0.\bar{3}$, $\frac{1}{6} = 0.6666\ldots = 0.\bar{6}$ and

repeating remainders

We can write a repeating decimal by placing a bar (called a *vinculum*) over the repeating digits in the decimal.

Therefore, we can write $\frac{1}{3} = 0.3333... = 0.\overline{3}$, $\frac{1}{6} = 0.6666... = 0.\overline{6}$ and

$\frac{4}{11} = 0.363636... = 0.\overline{36}$. We can write any rational number as either a terminating decimal or a repeating decimal.

To convert a repeating decimal to a fraction, follow the steps.

1. Multiply the repeating decimal by a power of ten so that you can write the repeating digit or digits of the decimal as a whole number.
2. Subtract the original decimal from the new decimal.
3. Solve the equation and find the fraction.

For example, let us convert the repeating decimal $0.\overline{3}$ to a fraction.

If $x = 0.\overline{3} = 0.333...$ then we can write

$$\begin{array}{rl} 10 \cdot x &= 3.333... \\ - \quad x &= 0.333... \\ \hline \end{array}$$

$9x = 3 \Rightarrow x = \frac{3}{9} = \frac{1}{3}$. Therefore, $0.\overline{3} = \frac{1}{3}$.

We can write a repeating decimal as a fraction more easily by following the steps.

1. Write the number without the decimal point.
2. Subtract the non-repeating part from this number.
3. Write the difference as the numerator of the fraction.
4. Write the denominator as a sequence of 9's and zeros. The number of 9's is as many as the number of repeating digits of the fraction part. The number of zeros is as many as the number of non-repeating digits of the fraction part.
5. Simplify the fraction.

number without decimal point

non-repeating part

a. $2.1\overline{35} = \frac{2135 - 21}{990} = \frac{2114}{990} = 2\frac{134}{990} = 2\frac{67}{450}$

two digits repeat in the fraction part

one digit does not repeat in the fraction part

b. $0.\overline{7} = \frac{7 - 0}{9} = \frac{7}{9}$

c. $0.\overline{45} = \frac{45 - 0}{99} = \frac{45}{99} = \frac{5}{11}$

d. $4.3\overline{2} = 3\frac{432 - 43}{90} = 3\frac{389}{90} = 4\frac{29}{90}$

e. $1.07\overline{3} = \frac{1073 - 107}{900} = \frac{966}{900} = \frac{161}{150} = 1\frac{11}{150}$

Repeating Decimal Numbers

Example 1 Show that each of the numbers (a) $1.323232\cdots = 1.\overline{32}$ and (b) $0.3405405405\ldots = 0.3\overline{405}$ is a rational number by expressing it as a quotient of two integers.

$131/99.$

$63/185.$

Intervals

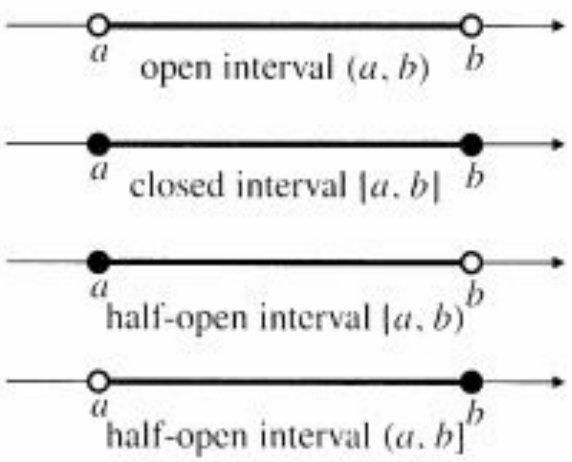


Figure P.2 Finite intervals

A subset of the real line is called an **interval** if it contains at least two numbers and also contains all real numbers between any two of its elements. For example, the set of real numbers x such that $x > 6$ is an interval, but the set of real numbers y such that $y \neq 0$ is not an interval. (Why?) It consists of two intervals.

If a and b are real numbers and $a < b$, we often refer to

(i) the **open interval** from a to b, denoted by (a, b), consisting of all real numbers x satisfying $a < x < b$.

(ii) the **closed interval** from a to b, denoted by $[a, b]$, consisting of all real numbers x satisfying $a \le x \le b$.

(iii) the **half-open interval** $[a, b)$, consisting of all real numbers x satisfying the inequalities $a \le x < b$.

(iv) the **half-open interval** $(a, b]$, consisting of all real numbers x satisfying the inequalities $a < x \le b$.

Solving of Inequalities

(a) $2x - 1 > x + 3$

(b) $-\frac{x}{3} \geq 2x - 1$

(c) $\frac{2}{x-1} \geq 5$

Solution

(a) $2x - 1 > x + 3$ Add 1 to both sides.

$2x > x + 4$ Subtract x from both sides.

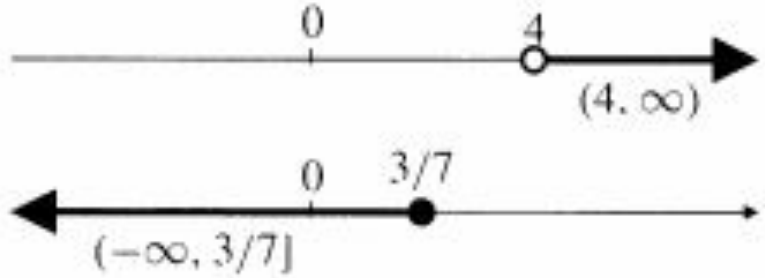
$x > 4$ The solution set is the interval $(4, \infty)$.

(b) $-\dfrac{x}{3} \geq 2x - 1$ Multiply both sides by -3.

$x \leq -6x + 3$ Add $6x$ to both sides.

$7x \leq 3$ Divide both sides by 7.

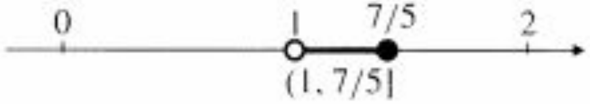
$x \leq \dfrac{3}{7}$ The solution set is the interval $(-\infty, 3/7]$.



(c) We transpose the 5 to the left side and simplify to rewrite the given inequality in an equivalent form:

$$\frac{2}{x-1} - 5 \geq 0 \quad \iff \quad \frac{2-5(x-1)}{x-1} \geq 0 \quad \iff \quad \frac{7-5x}{x-1} \geq 0.$$

The fraction $\dfrac{7-5x}{x-1}$ is undefined at $x = 1$ and is 0 at $x = 7/5$. Between these numbers it is positive if the numerator and denominator have the same sign, and negative if they have opposite sign. It is easiest to organize this sign information in a chart:



x		1		$7/5$	
$7-5x$	$+$	$+$	$+$	0	$-$
$x-1$	$-$	0	$+$	$+$	$+$
$(7-5x)/(x-1)$	$-$	undef	$+$	0	$-$

Thus the solution set of the given inequality is the interval $(1, 7/5]$.

Example 3 Solve the systems of inequalities:

(a) $3 \leq 2x + 1 \leq 5$ (b) $3x - 1 < 5x + 3 \leq 2x + 15$.

Solution

(a) Using the technique of Example 2, we can solve the inequality $3 \leq 2x + 1$ to get $2 \leq 2x$, so $x \geq 1$. Similarly, the inequality $2x + 1 \leq 5$ leads to $2x \leq 4$, so $x \leq 2$. The solution set of system (a) is therefore the closed interval $[1, 2]$.

(b) We solve both inequalities as follows:

$$\left.\begin{aligned} 3x - 1 &< 5x + 3 \\ -1 - 3 &< 5x - 3x \\ -4 &< 2x \\ -2 &< x \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} 5x + 3 &\leq 2x + 15 \\ 5x - 2x &\leq 15 - 3 \\ 3x &\leq 12 \\ x &\leq 4 \end{aligned}\right.$$

The solution set is the interval $(-2, 4]$.

Solve each compound inequality. Graph each solution.

a. $-2 + x + 1 < 3$

b. $-7 \leq 3x + 2 < 8$

c. $-6 < 2x - 3 \leq 5$

d. $-5 \leq 4 - 3x < 2$

e. $2 < 1 - \frac{x}{3} < 3$

f. $0 < \frac{3x + 2}{3} < -\frac{1}{3}$

g. $4 + 3 < 5x + 7 < x + 8$

h. $\frac{1}{2} \leq \frac{x + 1}{3} < \frac{3x}{4}$

Example 4 **Quadratic inequalities**

Solve: (a) $x^2 - 5x + 6 < 0$ (b) $2x^2 + 1 > 4x$.

Solve the inequality $\dfrac{3}{x-1} < -\dfrac{2}{x}$ and graph the solution set.

Solution We would like to multiply by $x(x-1)$ to clear the inequality of fractions, but this would require considering three cases separately. (What are they?) Instead, we will transpose and combine the two fractions into a single one:

$$\frac{3}{x-1} < -\frac{2}{x} \quad \iff \quad \frac{3}{x-1} + \frac{2}{x} < 0 \quad \iff \quad \frac{5x-2}{x(x-1)} < 0.$$

We examine the signs of the three factors in the left fraction to determine where that fraction is negative:

x		0		$2/5$		1	
$5x-2$	$-$	$-$	$-$	0	$+$	$+$	$+$
x	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$\dfrac{5x-2}{x(x-1)}$	$-$	undef	$+$	0	$-$	undef	$+$

The solution set of the given inequality is the union of these two intervals, namely, $(-\infty, 0) \cup (2/5, 1)$. See Figure P.5.

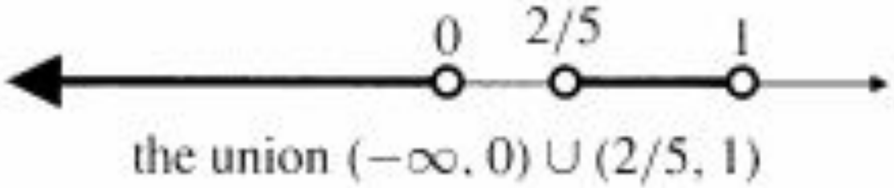


Figure P.5 The solution set for Example 5

[1] How do we know that $\sqrt{2}$ is an irrational number? Suppose, to the contrary, that $\sqrt{2}$ is rational. Then $\sqrt{2} = m/n$, where m and n are integers and $n \neq 0$. We can assume that the fraction m/n has been "reduced to lowest terms"; any common factors have been cancelled out. Now $m^2/n^2 = 2$, so $m^2 = 2n^2$, which is an even integer. Hence m must also be even. (The square of an odd integer is always odd.) Since m is even, we can write $m = 2k$, where k is an integer. Thus $4k^2 = 2n^2$ and $n^2 = 2k^2$, which is even. Thus n is also even. This contradicts the assumption that $\sqrt{2}$ could be written as a fraction m/n in lowest terms; m and n cannot both be even. Accordingly, there can be no rational number whose square is 2.

The Absolute Value

The **absolute value**, or **magnitude**, of a number x, denoted $|x|$ (read "the absolute value of x"), is defined by the formula

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

The vertical lines in the symbol $|x|$ are called **absolute value bars**.

$$|3| = 3, \quad |0| = 0, \quad |-5| = 5.$$

CONFUSING

Note that $|x| \geq 0$ for every real number x, and $|x| = 0$ only if $x = 0$. People sometimes find it confusing to say that $|x| = -x$ when x is negative, but this is correct since $-x$ is positive in that case. The symbol $\sqrt{a}$ always denotes the *nonnegative* square root of a, so an alternative definition of $|x|$ is $|x| = \sqrt{x^2}$.

Geometrically, $|x|$ represents the (nonnegative) distance from x to 0 on the real line. More generally, $|x - y|$ represents the (nonnegative) distance between the points x and y on the real line, since this distance is the same as that from the point $x - y$ to 0 (see Figure P.6):

$$|x - y| = \begin{cases} x - y, & \text{if } x \geq y \\ y - x, & \text{if } x < y. \end{cases}$$

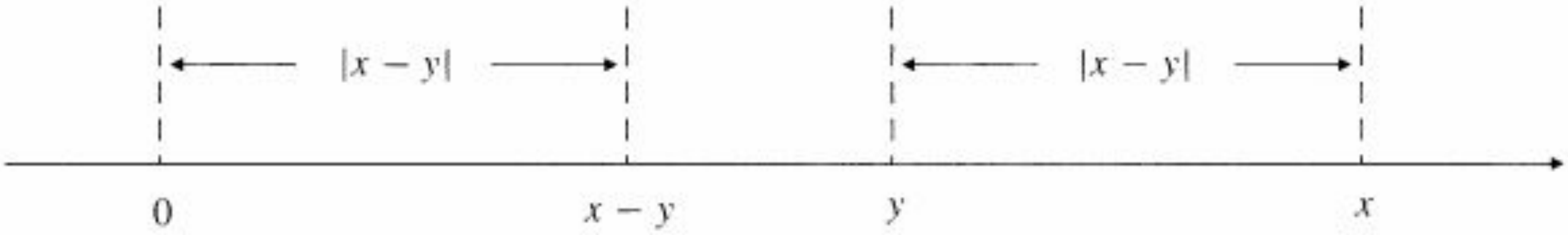


Figure P.6
$|x - y|$ = distance from x to y

Equations and Inequalities Involving Absolute Values

The equation $|x| = D$ (where $D > 0$) has two solutions, $x = D$ and $x = -D$: the two points on the real line that lie at distance D from the origin. Equations and inequalities involving absolute values can be solved algebraically by breaking them into cases according to the definition of absolute value, but often they can also be solved geometrically by interpreting absolute values as distances. For example, the inequality $|x - a| < D$ says that the distance from x to a is less than D, so x must lie between $a - D$ and $a + D$. (Or, equivalently, a must lie between $x - D$ and $x + D$.) If D is a positive number, then

$$\begin{aligned}
|x| = D \qquad &\iff \qquad \text{either } x = -D \text{ or } x = D\\
|x| < D \qquad &\iff \qquad -D < x < D\\
|x| \le D \qquad &\iff \qquad -D \le x \le D\\
|x| > D \qquad &\iff \qquad \text{either } x < -D \text{ or } x > D
\end{aligned}$$

More generally,

$$\begin{aligned}
|x - a| = D \qquad &\iff \qquad \text{either } x = a - D \text{ or } x = a + D\\
|x - a| < D \qquad &\iff \qquad a - D < x < a + D\\
|x - a| \le D \qquad &\iff \qquad a - D \le x \le a + D\\
|x - a| > D \qquad &\iff \qquad \text{either } x < a - D \text{ or } x > a + D
\end{aligned}$$

Solve: (a) $|2x + 5| = 3$ (b) $|3x - 2| \leq 1$.

Solution

(a) $|2x+5| = 3 \iff 2x+5 = \pm 3$. Thus, either $2x = -3-5 = -8$ or $2x = 3-5 = -2$. The solutions are $x = -4$ and $x = -1$.

(b) $|3x-2| \le 1 \iff -1 \le 3x-2 \le 1$. We solve this pair of inequalities:

$$\left\{\begin{aligned} -1 &\le 3x - 2 \\ -1+2 &\le 3x \\ 1/3 &\le x \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} 3x - 2 &\le 1 \\ 3x &\le 1+2 \\ x &\le 1 \end{aligned}\right\}$$

Thus the solutions lie in the interval $[1/3, 1]$.

Solve the equation $|x + 1| = |x - 3|$.

Solution The equation says that x is equidistant from -1 and 3. Therefore, x is the point halfway between -1 and 3; $x = (-1 + 3)/2 = 1$. Alternatively, the given equation says that either $x + 1 = x - 3$ or $x + 1 = -(x - 3)$. The first of these equations has no solutions; the second has the solution $x = 1$.

∎

What values of x satisfy the inequality $\left|5 - \frac{2}{x}\right| < 3$?

Solution We have

$$
\begin{aligned}
\left|5-\frac{2}{x}\right|<3 \quad \Longleftrightarrow \quad & -3<5-\frac{2}{x}<3 && \text{Subtract 5 from each member.} \\
& -8<-\frac{2}{x}<-2 && \text{Divide each member by } -2. \\
& 4>\frac{1}{x}>1 && \text{Take reciprocals.} \\
& \frac{1}{4}<x<1
\end{aligned}
$$

In Exercises 1–2, express the given rational number as a repeating decimal. Use a bar to indicate the repeating digits.

1. $\dfrac{2}{9}$ **2.** $\dfrac{1}{11}$

In Exercises 3–4, express the given repeating decimal as a quotient of integers in lowest terms.

3. $0.\overline{12}$ **4.** $3.2\overline{7}$

5. Express the rational numbers $1/7$, $2/7$, $3/7$, and $4/7$ as repeating decimals. (Use a calculator to give as many decimal digits as possible.) Do you see a pattern? Guess the decimal expansions of $5/7$ and $6/7$ and check your guesses.

6. Can two different decimals represent the same number? What number is represented by $0.999\ldots = 0.\overline{9}$?

In Exercises 7–12, express the set of all real numbers x satisfying the given conditions as an interval or a union of intervals.

7. $x \ge 0$ and $x \le 5$ **8.** $x < 2$ and $x \ge -3$

9. $x > -5$ or $x < -6$ **10.** $x \le -1$

11. $x > -2$ **12.** $x < 4$ or $x \ge 2$

In Exercises 13–26, solve the given inequality, giving the solution set as an interval or union of intervals.

13. $-2x > 4$ **14.** $3x + 5 \le 8$

15. $5x - 3 \le 7 - 3x$ **16.** $\dfrac{6-x}{4} \ge \dfrac{3x-4}{2}$

17. $3(2-x) < 2(3+x)$ **18.** $x^2 < 9$

19. $\dfrac{1}{2-x} < 3$ **20.** $\dfrac{x+1}{x} \ge 2$

21. $x^2 - 2x \le 0$ **22.** $6x^2 - 5x \le -1$

23. $x^3 > 4x$ **24.** $x^2 - x \le 2$

25. $\dfrac{x}{2} \ge 1 + \dfrac{4}{x}$ **26.** $\dfrac{3}{x-1} < \dfrac{2}{x+1}$

Solve the equations in Exercises 27–32.

27. $|x| = 3$ **28.** $|x - 3| = 7$

29. $|2t + 5| = 4$ **30.** $|1 - t| = 1$

31. $|8 - 3s| = 9$ **32.** $\left|\dfrac{s}{2} - 1\right| = 1$

In Exercises 33–40, write the interval defined by the given inequality.

33. $|x| < 2$ **34.** $|x| \le 2$

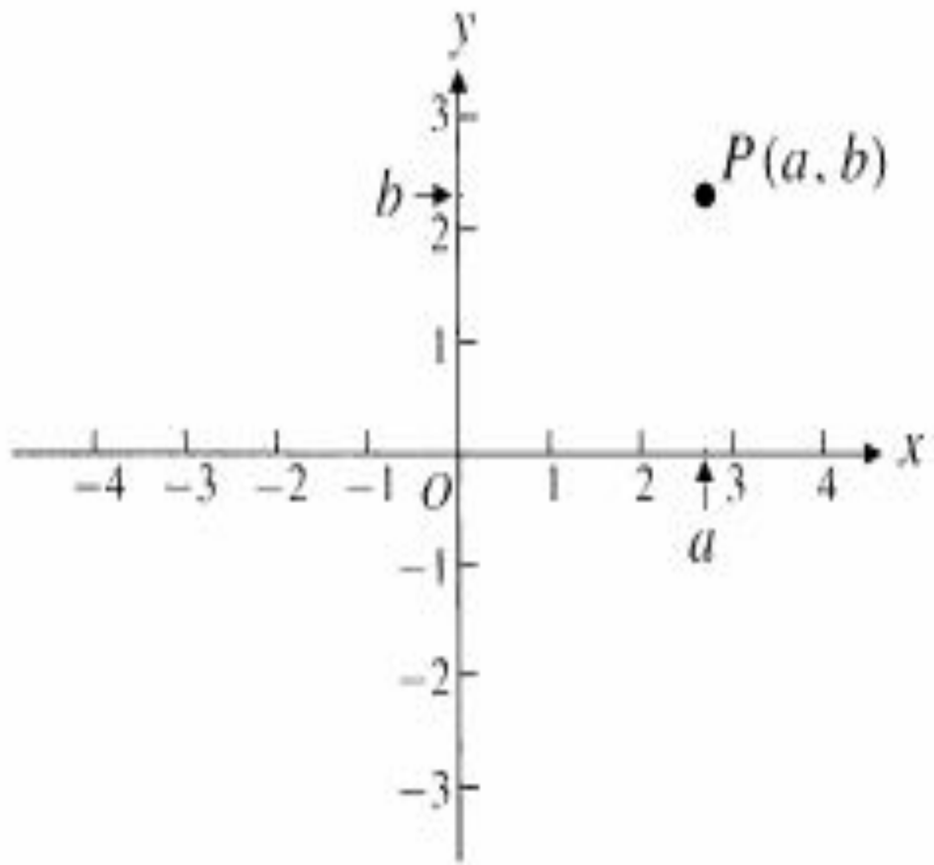
35. $|s - 1| \le 2$ **36.** $|t + 2| < 1$

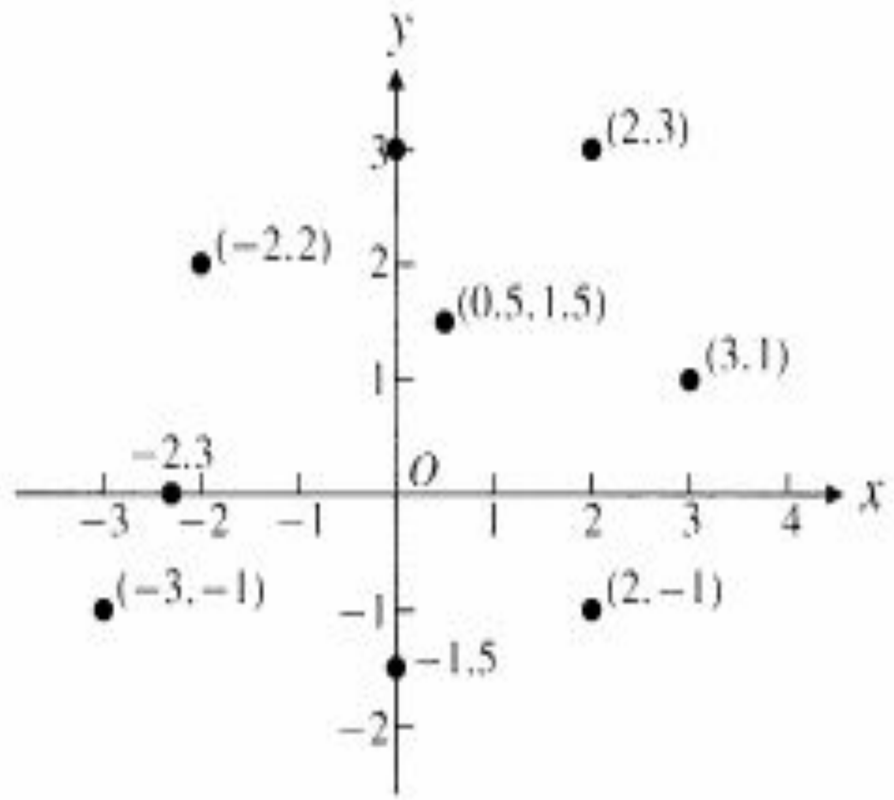
37. $|3x - 7| < 2$ **38.** $|2x + 5| < 1$

39. $\left|\dfrac{x}{2} - 1\right| \le 1$ **40.** $\left|2 - \dfrac{x}{2}\right| < \dfrac{1}{2}$

Cartesian Coordinate Plane

The positions of all points in a plane can be measured with respect to two perpendicular real lines in the plane intersecting at the 0-point of each. These lines are called **coordinate axes** in the plane. Usually (but not always) we call one of these axes the x-axis and draw it horizontally with numbers x on it increasing to the right; then we call the other the y-axis, and draw it vertically with numbers y on it increasing upward. The point of intersection of the coordinate axes (the point where x and y are both zero) is called the **origin** and is often denoted by the letter O.


y
x
P(a, b)
b
a
O
3
2
1
−1
−2
−3
−4
−3
−2
−1
1
2
3
4


y
x
O
(2,3)
(−2,2)
(0.5,1.5)
(3,1)
−2.3
(−3,−1)
(2,−1)
−1.5
3
2
1
−1
−2
−3
−2
−1
1
2
3
4

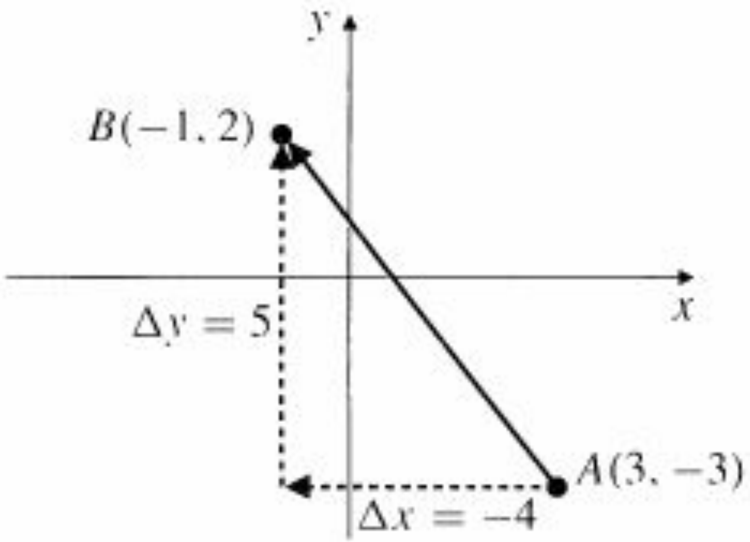
Increments and Distances

When a particle moves from one point to another, the net changes in its coordinates are called increments. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point. An **increment** in a variable is the net change in the value of the variable. If x changes from x_1 to x_2, then the increment in x is $\Delta x = x_2 - x_1$.

Example 1 Find the increments in the coordinates of a particle that moves from $A(3, -3)$ to $B(-1, 2)$.

Solution The increments (see Figure P.11) are:

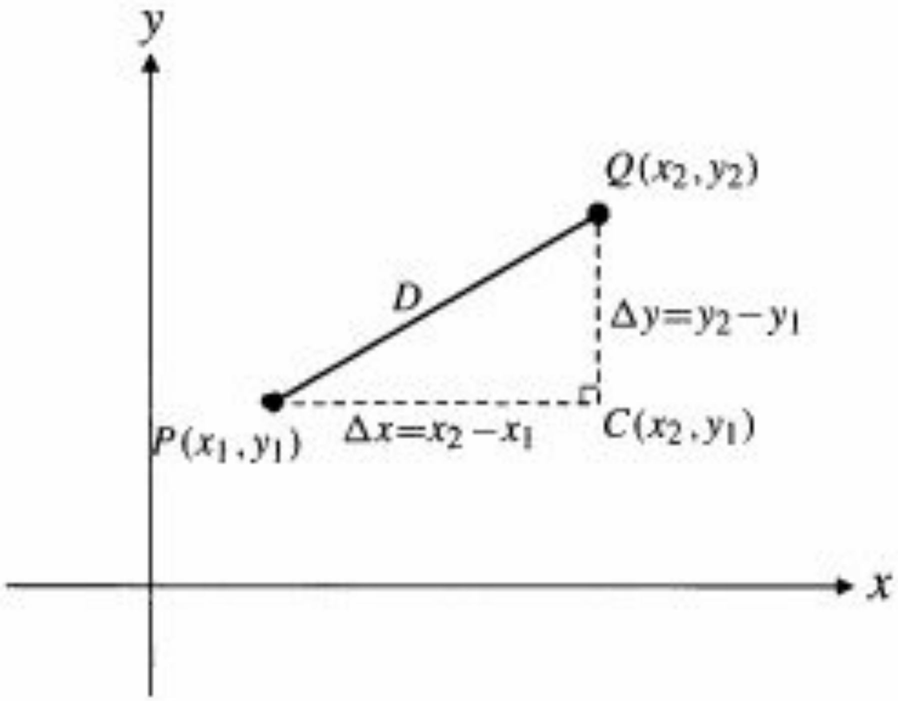
$$\Delta x = -1 - 3 = -4 \qquad \text{and} \qquad \Delta y = 2 - (-3) = 5.$$



Distance formula for points in the plane

The distance D between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



Example 2 The distance between $A(3, -3)$ and $B(-1, 2)$ in Figure P.11 is

$$\sqrt{(-1-3)^2 + (2-(-3))^2} = \sqrt{(-4)^2 + 5^2} = \sqrt{41} \text{ units.}$$

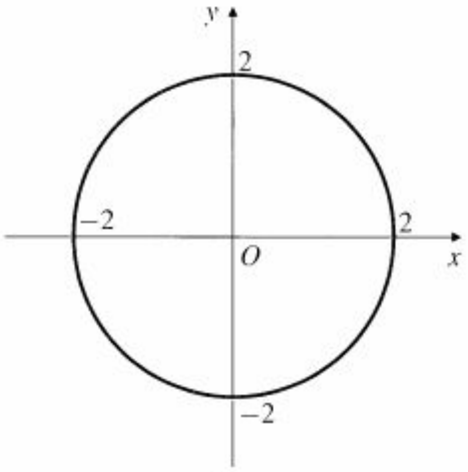
Example 3 The distance from the origin $O(0, 0)$ to a point $P(x, y)$ is

$$\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}.$$

Graphs

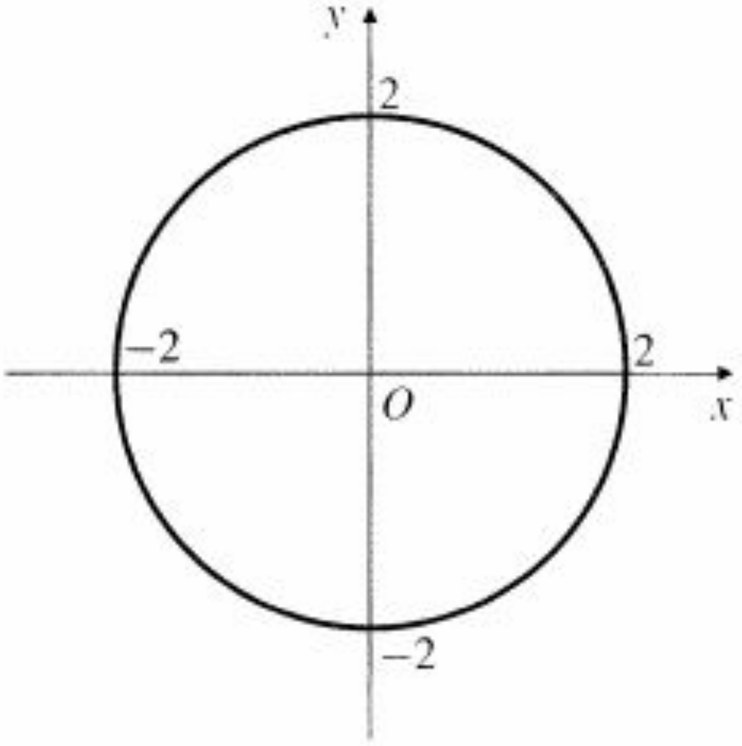
The **graph** of an equation (or inequality) involving the variables x and y is the set of all points $P(x, y)$ whose coordinates satisfy the equation (or inequality).

Example 4 The equation $x^2 + y^2 = 4$ represents all points $P(x, y)$ whose distance from the origin is $\sqrt{x^2 + y^2} = \sqrt{4} = 2$. These points lie on the **circle** of radius 2 centred at the origin. This circle is the graph of the equation $x^2 + y^2 = 4$. (See Figure P.13(a).)



(a)

Example 5 Points (x, y) whose coordinates satisfy the inequality $x^2 + y^2 \leq 4$ all have distance ≤ 2 from the origin. The graph of the inequality is therefore the disk of radius 2 centred at the origin. (See Figure P.13(b).)



(b)

Example 6 Consider the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, and $(-2, 4)$. These points (and all others satisfying the equation) lie on a smooth curve called a **parabola**. (See Figure P.14.)

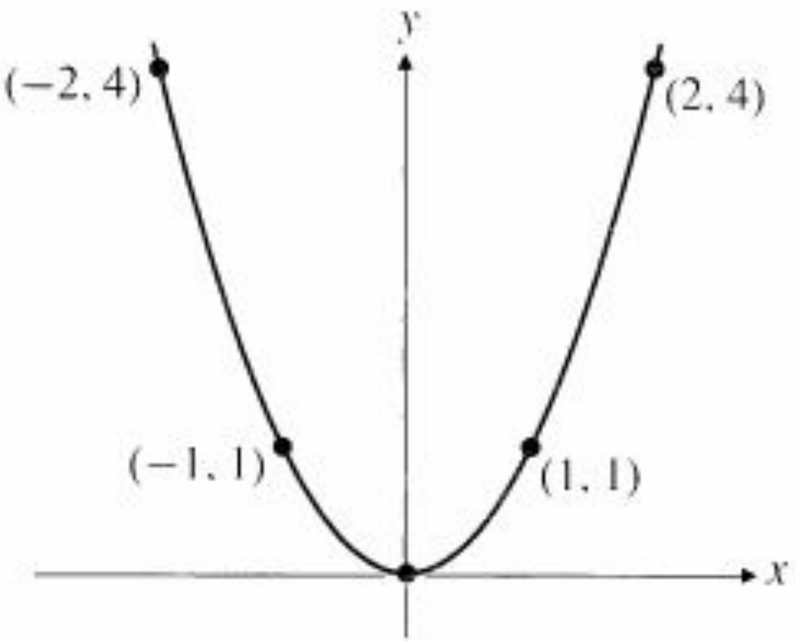


Figure P.14 The parabola $y = x^2$

Straight Lines

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$, respectively, the **run** and the **rise** between P_1 and P_2. Two such points always determine a unique **straight line** (usually called simply a **line**) passing through them both. We call the line $P_1 P_2$.

Any nonvertical line in the plane has the property that the ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

has the *same value* for every choice of two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line. (See Figure P.15.) The constant $m = \Delta y / \Delta x$ is called the **slope** of the nonvertical line.

SLOPE

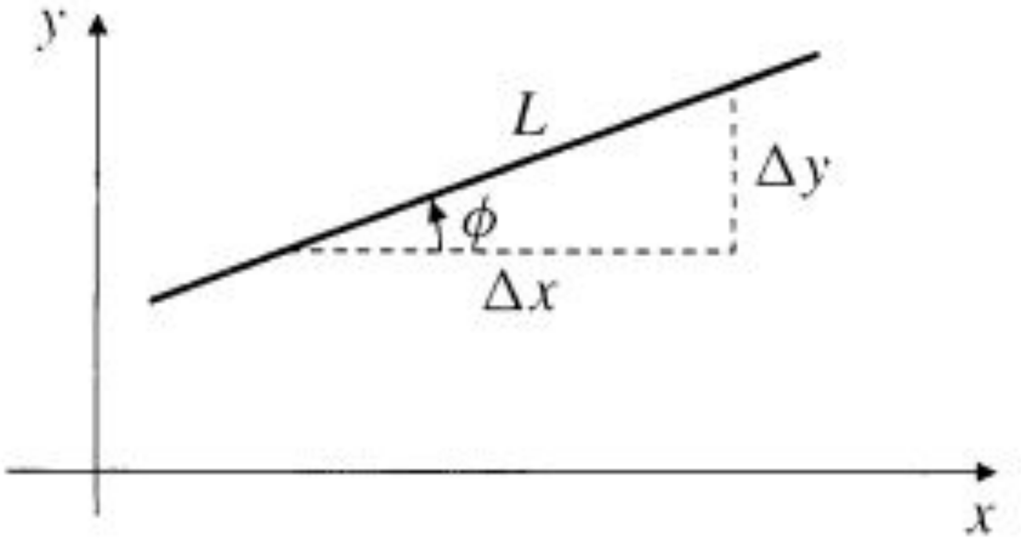


Figure P.16 Line L has inclination ϕ

The slope tells us the direction and steepness of a line. A line with positive slope rises uphill to the right; one with negative slope falls downhill to the right. The greater the absolute value of the slope, the steeper the rise or fall. Since the run Δx is zero for a vertical line, we cannot form the ratio m; the slope of a vertical line is *undefined*.

The direction of a line can also be measured by an angle. The **inclination** of a line is the smallest counterclockwise angle from the positive direction of the x-axis to the line. In Figure P.16 the angle ϕ (the Greek letter "phi") is the inclination of the line L. The inclination ϕ of any line satisfies $0^\circ \le \phi < 180^\circ$. The inclination of a horizontal line is 0° and that of a vertical line is 90°.

Provided equal scales are used on the coordinate axes, the relationship between the slope m of a nonvertical line and its inclination ϕ is shown in Figure P.16:

$$m = \frac{\Delta y}{\Delta x} = \tan \phi.$$

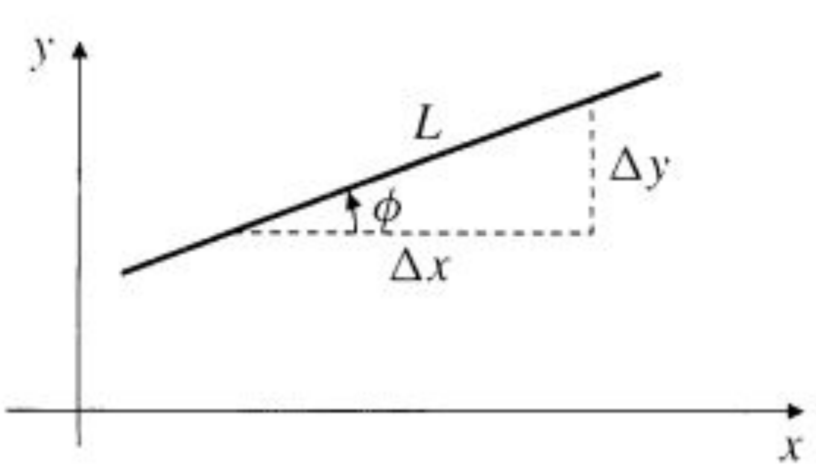
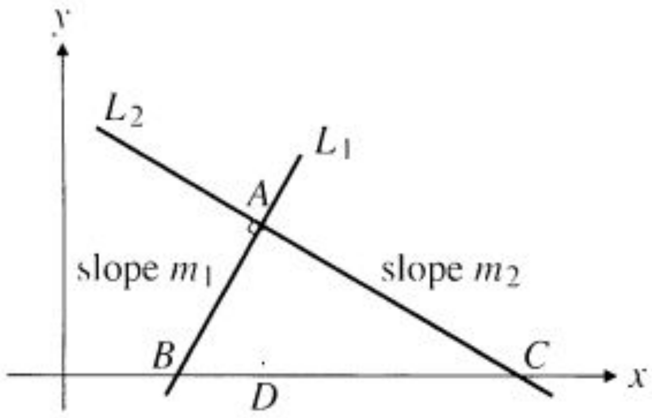


Figure P.16 Line L has inclination ϕ



Parallel lines have the same inclination. If they are not vertical, they must therefore have the same slope. Conversely, lines with equal slopes have the same inclination and so are parallel.

If two nonvertical lines, L_1 and L_2, are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2} \qquad \text{and} \qquad m_2 = -\frac{1}{m_1}.$$

Equations of Lines

Straight lines are particularly simple graphs, and their corresponding equations are also simple. All points on the vertical line through the point a on the x-axis have their x-coordinates equal to a. Thus $x = a$ is the equation of the line. Similarly, $y = b$ is the equation of the horizontal line meeting the y-axis at b.

Example 8 The horizontal and vertical lines passing through the point $(3, 1)$ (Figure P.18) have equations $y = 1$ and $x = 3$, respectively. ■

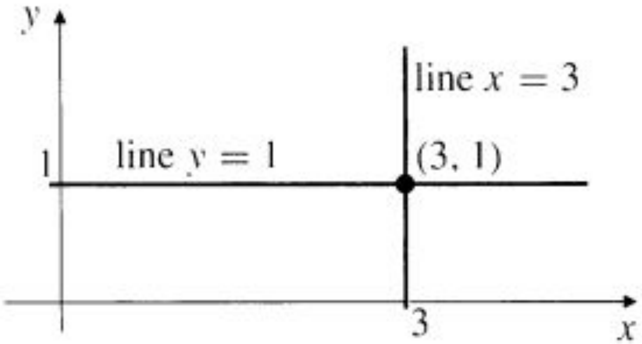


Figure P.18 The lines $y = 1$ and $x = 3$

The equation

$$y = m(x - x_1) + y_1$$

is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m.

Find an equation of the line of slope -2 through the point $(1, 4)$.

Solution We substitute $x_1 = 1$, $y_1 = 4$, and $m = -2$ into the point-slope form of the equation and obtain

$$y = -2(x - 1) + 4 \qquad \text{or} \qquad y = -2x + 6.$$

Find an equation of the line through the points $(1, -1)$ and $(3, 5)$.

Solution The slope of the line is $m = \dfrac{5-(-1)}{3-1} = 3$. We can use this slope with either of the two points to write an equation of the line. If we use $(1, -1)$ we get

$$y = 3(x-1) - 1, \qquad \text{which simplifies to} \quad y = 3x - 4.$$

If we use $(3, 5)$ we get

$$y = 3(x-3) + 5, \qquad \text{which also simplifies to} \quad y = 3x - 4.$$

Either way, $y = 3x - 4$ is an equation of the line.

INTERCEPTED POINTS

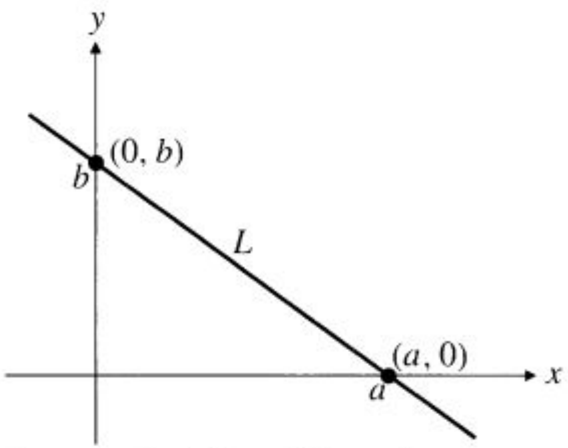


Figure P.19 Line L has x-intercept a and y-intercept b

The y-coordinate of the point where a nonvertical line intersects the y-axis is called the ***y*-intercept** of the line. (See Figure P.19.) Similarly, the ***x*-intercept** of a nonhorizontal line is the x-coordinate of the point where it crosses the x-axis. A line with slope m and y-intercept b passes through the point $(0, b)$, so its equation is

$$y = m(x - 0) + b \qquad \text{or, more simply,} \qquad y = mx + b.$$

Find the slope and the two intercepts of the line with equation $8x + 5y = 20.$

Solution Solving the equation for y we get

$$y = \frac{20 - 8x}{5} = -\frac{8}{5}x + 4.$$

Comparing this with the general form $y = mx + b$ of the slope–y-intercept equation, we see that the slope of the line is $m = -8/5$, and the y-intercept is $b = 4$.

To find the x-intercept put $y = 0$ and solve for x, obtaining $8x = 20$, or $x = 5/2$. The x-intercept is $a = 5/2$.

Example 12 The relationship between Fahrenheit temperature (F) and Celsius temperature (C) is given by a linear equation of the form $F = mC + b$. The freezing point of water is $F = 32°$ or $C = 0°$, while the boiling point is $F = 212°$ or $C = 100°$. Thus

$$32 = 0m + b \qquad \text{and} \qquad 212 = 100m + b,$$

so $b = 32$ and $m = (212 - 32)/100 = 9/5$. The relationship is given by the linear equation

$$F = \frac{9}{5}C + 32 \qquad \text{or} \qquad C = \frac{5}{9}(F - 32).$$

Describe the graphs of the equations and inequalities in Exercises

$$x^2 + y^2 = 1$$

$$x^2 + y^2 \leq 1$$

$$y \geq x^2$$

Find the point of intersection of the lines $3x + 4y = -6$ and $2x - 3y = 13$.

Find the point of intersection of the lines $2x + y = 8$ and $5x - 7y = 1$.

A line passes through $(-2, 5)$ and $(k, 1)$ and has x-intercept 3. Find k.

The cost of printing x copies of a pamphlet is $\$C$, where $C = Ax + B$ for certain constants A and B. If it costs \$5,000 to print 10,000 copies and \$6,000 to print 15,000 copies, how much will it cost to print 100,000 copies?

For what value of k is the line $2x + ky = 3$ perpendicular to the line $4x + y = 1$? For what value of k are the lines parallel?

Find the line that passes through the point $(1, 2)$ and through the point of intersection of the two lines $x + 2y = 3$ and $2x - 3y = -1$.

By calculating the lengths of its three sides, show that the triangle with vertices at the points $A(2, 1)$, $B(6, 4)$, and $C(5, -3)$ is isosceles.

Show that the triangle with vertices $A(0, 0)$, $B(1, \sqrt{3})$, and $C(2, 0)$ is equilateral.

Show that the points $A(2, -1)$, $B(1, 3)$, and $C(-3, 2)$ are three vertices of a square and find the fourth vertex.

Find the coordinates of the midpoint on the line segment P_1P_2 joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Find the coordinates of the point of the line segment joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ that is two-thirds of the way from P_1 to P_2.

The point P lies on the x-axis and the point Q lies on the line $y = -2x$. The point $(2, 1)$ is the midpoint of PQ. Find the coordinates of P.

By calculating the lengths of its three sides, show that the triangle with vertices at the points $A(2, 1)$, $B(6, 4)$, and $C(5, -3)$ is isosceles.

For what value of k is the line $2x + ky = 3$ perpendicular to the line $4x + y = 1$? For what value of k are the lines parallel?

Find the line that passes through the point $(1, 2)$ and through the point of intersection of the two lines $x + 2y = 3$ and $2x - 3y = -1$.

Circles and Disks

The **circle** having **centre** C and **radius** a is the set of all points in the plane that are at distance a from the point C.

The distance from $P(x, y)$ to the point $C(h, k)$ is $\sqrt{(x-h)^2 + (y-k)^2}$, so that the equation of the circle of radius $a > 0$ with centre at $C(h, k)$ is

Standard equation of a circle

The circle with centre (h, k) and radius $a > 0$ has equation

$$(x-h)^2 + (y-k)^2 = a^2.$$

In particular, the circle with centre at the origin $(0, 0)$ and radius a has equation

$$x^2 + y^2 = a^2.$$

Example 1 The circle with radius 2 and centre (1, 3) (Figure P.20) has equation

$$(x-1)^2 + (y-3)^2 = 4.$$

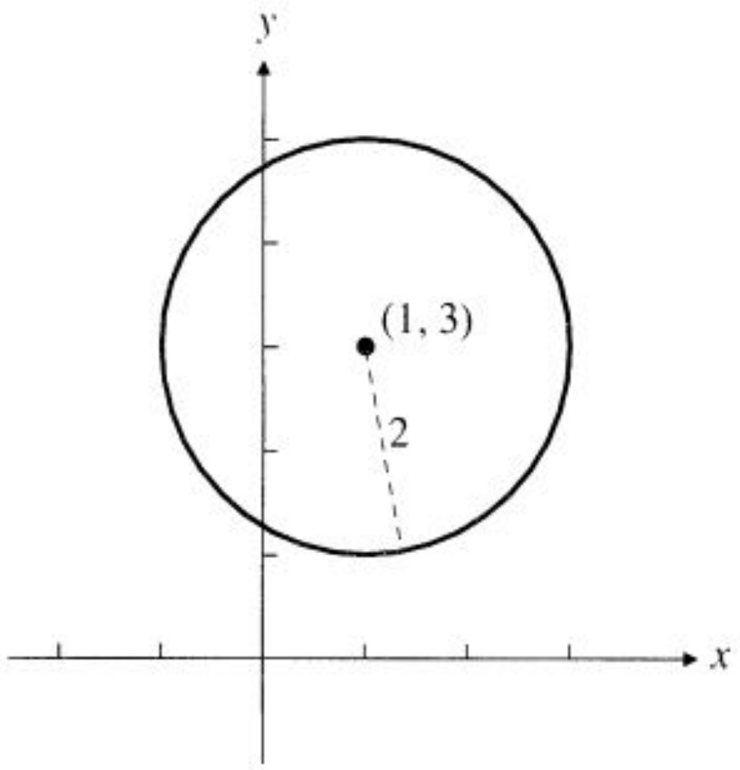


Figure P.20 Circle
$(x-1)^2 + (y-3)^2 = 4$

Find the centre and radius of the circle $x^2 + y^2 - 4x + 6y = 3$.

Solution Observe that x^2-4x are the first two terms of the binomial square $(x-2)^2 = x^2 - 4x + 4$, and $y^2 + 6y$ are the first two terms of the square $(y+3)^2 = y^2 + 6y + 9$. Hence we add $4 + 9$ to both sides of the given equation and obtain

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 3 + 4 + 9 \quad \text{or} \quad (x-2)^2 + (y+3)^2 = 16.$$

This is the equation of a circle with centre $(2, -3)$ and radius 4.

Functions and Their Graphs

The area of a circle depends on its radius. The temperature at which water boils depends on the altitude above sea level. The interest paid on a cash investment depends on the length of time for which the investment is made.

Whenever one quantity depends on another quantity, we say that the former quantity is a function of the latter. For instance, the area A of a circle depends on the radius r according to the formula

$$A = \pi r^2,$$

A **function** f on a set D into a set S is a rule that assigns a *unique* element $f(x)$ in S to each element x in D.

In this definition $D = \mathcal{D}(f)$ (read "D of f") is the domain of the function f. The range $\mathcal{R}(f)$ of f is the subset of S consisting of all *values* $f(x)$ of the function. Think of a function f as a kind of machine (Figure P.35) that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain.

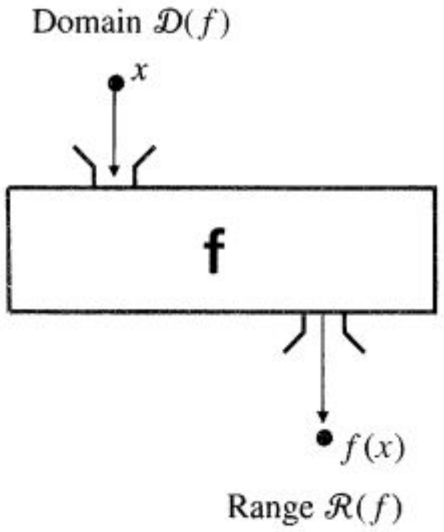


Figure P.35 A function machine

In this definition $D = \mathcal{D}(f)$ (read "D of f") is the domain of the function f. The range $\mathcal{R}(f)$ of f is the subset of S consisting of all *values* $f(x)$ of the function. Think of a function f as a kind of machine (Figure P.35) that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain.

There are several ways to represent a function symbolically. The squaring function that converts any input real number x into its square x^2 can be denoted:

(a) by a formula such as $y = x^2$, which uses a dependent variable y to denote the value of the function;

(b) by a formula such as $f(x) = x^2$, which defines a function symbol f to name the function; or

(c) by a mapping rule such as $x \longrightarrow x^2$. (Read this as "x goes to x^2.")

Example 1 The volume of a ball of radius r is given by the function

$$V(r) = \frac{4}{3}\pi r^3$$

for $r \geq 0$. Thus the volume of a ball of radius 3 ft is

$$V(3) = \frac{4}{3}\pi(3)^3 = 36\pi \text{ ft}^3.$$

Note how the variable r is replaced by the special value 3 in the formula defining the function to obtain the value of the function at $r = 3$. ∎

Example 2 A function F is defined for all real numbers t by

$$F(t) = 2t + 3.$$

Find the output values of F that correspond to the input values 0, 2, $x + 2$, and $F(2)$.

Solution In each case we substitute the given input for t in the definition of F:

$$\begin{aligned} F(0) &= 2(0) + 3 = 0 + 3 = 3 \\ F(2) &= 2(2) + 3 = 4 + 3 = 7 \\ F(x+2) &= 2(x+2) + 3 = 2x + 7 \\ F(F(2)) &= F(7) = 2(7) + 3 = 17. \end{aligned}$$

∎

The domain convention

When a function f is defined without specifying its domain, we assume that the domain consists of all real numbers x for which the value $f(x)$ of the function is a real number.

In practice, it is often easy to determine the domain of a function $f(x)$ given by an explicit formula. We just have to exclude those values of x that would result in dividing by 0 or taking even roots of negative numbers.

Example 3 **The square root function.** The domain of $f(x) = \sqrt{x}$ is the interval $[0, \infty)$, since negative numbers do not have real square roots. We have $f(0) = 0$, $f(4) = 2$, $f(10) \approx 3.16228$. Note that, although there are *two* numbers whose square is 4, namely, -2 and 2, only *one* of these numbers, 2, is the square root of 4. (Remember that a function assigns a *unique* value to each element in its domain; it cannot assign two different values to the same input.) The **square root function** $\sqrt{x}$ always denotes the *nonnegative* square root of x. The two solutions of the equation $x^2 = 4$ are $x = \sqrt{4} = 2$ and $x = -\sqrt{4} = -2$. ■

Example 4 The domain of the function $h(x) = \dfrac{x}{x^2 - 4}$ consists of all real numbers except $x = -2$ and $x = 2$. Expressed in terms of intervals,

$$\mathcal{D}(f) = (-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

Most of the functions we encounter will have domains that are either intervals or unions of intervals.

Example 5 The domain of $S(t) = \sqrt{1 - t^2}$ consists of all real numbers t for which $1 - t^2 \geq 0$. Thus we require that $t^2 \leq 1$, or $-1 \leq t \leq 1$. The domain is the closed interval $[-1, 1]$.

Graphs of Functions

An old maxim states that "a picture is worth a thousand words." This is certainly true in mathematics; the behaviour of a function is best described by drawing its graph.

The **graph of a function** f is just the graph of the *equation* $y = f(x)$. It consists of those points in the Cartesian plane whose coordinates (x, y) are pairs of input–output values for f. Thus (x, y) lies on the graph of f provided x is in the domain of f and $y = f(x)$.

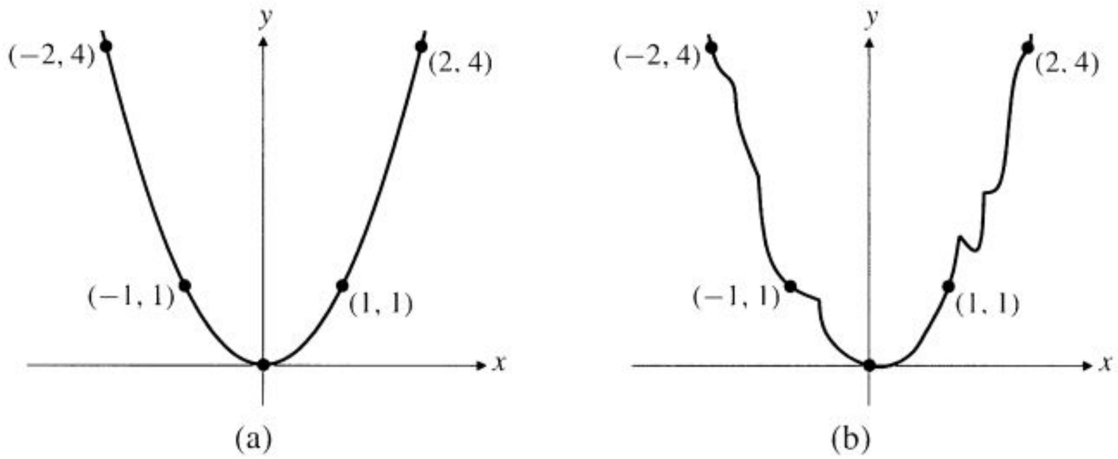
Drawing the graph of a function f sometimes involves making a table of coordinate pairs $(x, f(x))$ for various values of x in the domain of f, then plotting these points and connecting them with a "smooth curve."

Example 6 Graph the function $f(x) = x^2$.

Solution Make a table of (x, y) pairs that satisfy $y = x^2$. (See Table 1.) Now plot the points and join them with a smooth curve. (See Figure P.36(a).) ■

Table 1.

x	$y = f(x)$
-2	4
-1	1
0	0
1	1
2	4



(a) Correct graph of $f(x) = x^2$

(b) Incorrect graph of $f(x) = x^2$