

CHAPTER 7

FUNCTIONS



SECTION 7.1

Functions Defined on General Sets

Functions Defined on General Sets

We have already defined a function as a certain type of relation. The following is a restatement of the definition of function that includes additional terminology associated with the concept.

• Definition

A **function f from a set X to a set Y** , denoted $f: X \rightarrow Y$, is a relation from X , the **domain**, to Y , the **co-domain**, that satisfies two properties: (1) every element in X is related to some element in Y , and (2) no element in X is related to more than one element in Y . Thus, given any element x in X , there is a unique element in Y that is related to x by f . If we call this element y , then we say that “ f sends x to y ” or “ f maps x to y ” and write $x \xrightarrow{f} y$ or $f: x \rightarrow y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x** , or
the output of f for the input x , or
the value of f at x , or
the image of x under f .

The set of all values of f taken together is called the *range of f* or the *image of X under f* . Symbolically,

range of f = image of X under f = $\{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}$.

Given an element y in Y , there may exist elements in X with y as their image. If $f(x) = y$, then x is called a **preimage of y** or an **inverse image of y** . The set of all inverse images of y is called the *inverse image of y* . Symbolically,

the inverse image of y = $\{x \in X \mid f(x) = y\}$.



Arrow Diagrams

Arrow Diagrams

We have known that if X and Y are finite sets, you can define a function f from X to Y by drawing an arrow diagram.

You make a list of elements in X and a list of elements in Y , and draw an arrow from each element in X to the corresponding element in Y , as shown in Figure 7.1.1.

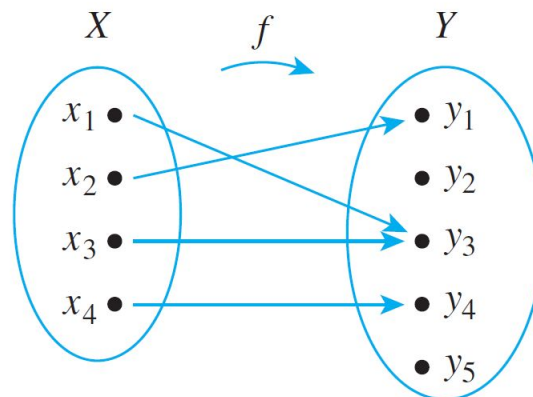


Figure 7.1.1



Arrow Diagrams

This arrow diagram does define a function because

1. Every element of X has an arrow coming out of it.
2. No element of X has two arrows coming out of it that point to two different elements of Y .

Example 2 – A Function Defined by an Arrow Diagram

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram in Figure 7.1.3.

- Write the domain and co-domain of f .
- Find $f(a)$, $f(b)$, and $f(c)$.
- What is the range of f ?
- Is c an inverse image of 2?
Is b an inverse image of 3?
- Find the inverse images of 2, 4, and 1.
- Represent f as a set of ordered pairs.

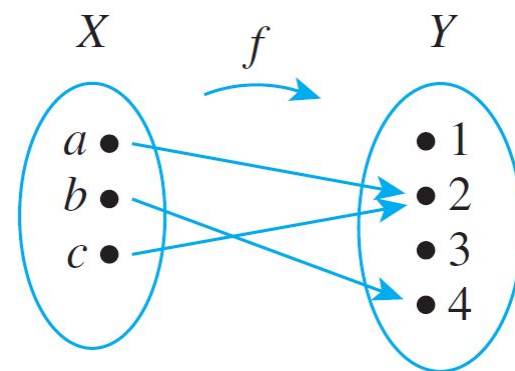



Figure 7.1.3



Example 2 – *Solution*

- a. domain of $f = \{a, b, c\}$, co-domain of $f = \{1, 2, 3, 4\}$
- b. $f(a) = 2, f(b) = 4, f(c) = 2$
- c. range of $f = \{2, 4\}$
- d. Yes, No
- e. inverse image of $2 = \{a, c\}$
inverse image of $4 = \{b\}$
inverse image of $1 = \emptyset$ (*since no arrows point to 1*)
- f. $\{(a, 2), (b, 4), (c, 2)\}$



Arrow Diagrams

In Example 2 there are no arrows pointing to the 1 or the 3.

This illustrates the fact that although each element of the domain of a function must have an arrow pointing out from it, there can be elements of the co-domain to which no arrows point.

Note also that there are two arrows pointing to the 2—one coming from a and the other from c .



Arrow Diagrams

Earlier we have given a test for determining whether two functions with the same domain and co-domain are equal, saying that the test results from the definition of a function as a binary relation.

We formalize this justification in Theorem 7.1.1.

Theorem 7.1.1 A Test for Function Equality

If $F: X \rightarrow Y$ and $G: X \rightarrow Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example 3 – *Equality of Functions*

- a. Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows: For all x in J_3 ,

$$f(x) = (x^2 + x + 1) \bmod 3 \quad \text{and} \quad g(x) = (x + 2)^2 \bmod 3.$$

Does $f = g$?

- b. Let $F: \mathbf{R} \rightarrow \mathbf{R}$ and $G: \mathbf{R} \rightarrow \mathbf{R}$ be functions. Define new functions $F + G: \mathbf{R} \rightarrow \mathbf{R}$ and $G + F: \mathbf{R} \rightarrow \mathbf{R}$ as follows: For all $x \in \mathbf{R}$,

$$(F + G)(x) = F(x) + G(x) \quad \text{and} \quad (G + F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

Example 3 – Solution

a. Yes, the table of values shows that $f(x) = g(x)$ for all x in J_3 .

x	$x^2 + x + 1$	$f(x) = (x^2 + x + 1) \bmod 3$	$(x + 2)^2$	$g(x) = (x + 2)^2 \bmod 3$
0	1	$1 \bmod 3 = 1$	4	$4 \bmod 3 = 1$
1	3	$3 \bmod 3 = 0$	9	$9 \bmod 3 = 0$
2	7	$7 \bmod 3 = 1$	16	$16 \bmod 3 = 1$

b. Again the answer is yes. For all real numbers x ,

$$(F + G)(x) = F(x) + G(x) \quad \text{by definition of } F + G$$

$$= G(x) + F(x) \quad \text{by the commutative law for addition of real numbers}$$

$$= (G + F)(x) \quad \text{by definition of } G + F$$

Hence $F + G = G + F$.



Examples of Functions

Example 4 – *The Identity Function on a Set*

Given a set X , define a function I_X from X to X by

$$I_X(x) = x \quad \text{for all } x \text{ in } X.$$

The function I_X is called the **identity function on X** because it sends each element of X to the element that is identical to it. Thus the identity function can be pictured as a machine that sends each piece of input directly to the output chute without changing it in any way.

Let X be any set and a_{ij}^k and $\phi(z)$ are elements of X . Find $I_X(a_{ij}^k)$ and $I_X(\phi(z))$.



Example 4 – *Solution*

Whatever is input to the identity function comes out unchanged, so $I_X(a_{ij}^k) = a_{ij}^k$ and $I_X(\phi(z)) = \phi(z)$.



Examples of Functions

- **Definition Logarithms and Logarithmic Functions**

Let b be a positive real number with $b \neq 1$. For each positive real number x , the **logarithm with base b of x** , written $\log_b x$, is the exponent to which b must be raised to obtain x . Symbolically,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base b** is the function from \mathbf{R}^+ to \mathbf{R} that takes each positive real number x to $\log_b x$.



Example 8 – *The Logarithmic Function with Base b*

Find the following:

a. $\log_3 9$

b. $\log_2 \left(\frac{1}{2} \right)$

c. $\log_{10}(1)$

d. $\log_2(2^m)$ (m is any real number)


e. $2^{\log_2 m}$ ($m > 0$)

Solution:

a. $\log_3 9 = 2$ because $3^2 = 9$.

b. $\log_2 \left(\frac{1}{2} \right) = -1$ because $2^{-1} = \frac{1}{2}$.

c. $\log_{10}(1) = 0$ because $10^0 = 1$.



Example 8 – *Solution*

cont'd

- d.** $\log_2(2^m) = m$ because the exponent to which 2 must be raised to obtain 2^m is m .

- e.** $2^{\log_2 m} = m$ because $\log_2 m$ is the exponent to which 2 must be raised to obtain m .



Examples of Functions

We have known that if S is a nonempty, finite set of characters, then a **string over S** is a finite sequence of elements of S .

The number of characters in a string is called the **length** of the string. The **null string over S** is the “string” with no characters.

It is usually denoted ϵ and is said to have length 0.



Example 9 – *Encoding and Decoding Functions*

Digital messages consist of finite sequences of 0's and 1's. When they are communicated across a transmission channel, they are frequently coded in special ways to reduce the possibility that they will be garbled by interfering noise in the transmission lines.

For example, suppose a message consists of a sequence of 0's and 1's. A simple way to encode the message is to write each bit three times. Thus the message

00101111

would be encoded as

000000111000111111111111.



Example 9 – *Encoding and Decoding Functions*

cont'd

The receiver of the message decodes it by replacing each section of three identical bits by the one bit to which all three are equal.

Let A be the set of all strings of 0's and 1's, and let T be the set of all strings of 0's and 1's that consist of consecutive triples of identical bits.

The encoding and decoding processes described above are actually functions from A to T and from T to A .



Example 9 – *Encoding and Decoding Functions*

cont'd

The encoding function E is the function from A to T defined as follows: For each string $s \in A$,

$E(s)$ = the string obtained from s by replacing each bit of s by the same bit written three times.

The decoding function D is defined as follows: For each string $t \in T$,

$D(t)$ = the string obtained from t by replacing each consecutive triple of three identical bits of t by a single copy of that bit.



Example 9 – *Encoding and Decoding Functions*

cont'd

The advantage of this particular coding scheme is that it makes it possible to do a certain amount of error correction when interference in the transmission channels has introduced errors into the stream of bits.

If the receiver of the coded message observes that one of the sections of three consecutive bits that should be identical does not consist of identical bits, then one bit differs from the other two.

In this case, if errors are rare, it is likely that the single bit that is different is the one in error, and this bit is changed to agree with the other two before decoding.



Boolean Functions



Boolean Functions

We have discussed earlier that how to find input/output tables for certain digital logic circuits.

Any such input/output table defines a function in the following way: The elements in the input column can be regarded as ordered tuples of 0's and 1's; the set of all such ordered tuples is the domain of the function.

The elements in the output column are all either 0 or 1; thus $\{0, 1\}$ is taken to be the co-domain of the function. The relationship is that which sends each input element to the output element in the same row.



Boolean Functions

- **Definition**

An (***n*-place**) **Boolean function** f is a function whose domain is the set of all ordered n -tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \rightarrow \{0, 1\}$.



Example 11 – A Boolean Function

Consider the three-place Boolean function defined from the set of all 3-tuples of 0's and 1's to $\{0, 1\}$ as follows: For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3) \text{ mod } 2.$$

Describe f using an input/output table.

Solution:

$$f(1, 1, 1) = (1 + 1 + 1) \text{ mod } 2 = 3 \text{ mod } 2 = 1$$

$$f(1, 1, 0) = (1 + 1 + 0) \text{ mod } 2 = 2 \text{ mod } 2 = 0$$

Example 11 – *Solution*

cont'd

The rest of the values of f can be calculated similarly to obtain the following table.

Input			Output
x_1	x_2	x_3	$(x_1 + x_2 + x_3) \bmod 2$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0



Checking Whether a Function Is Well Defined



Checking Whether a Function Is Well Defined

It can sometimes happen that what appears to be a function defined by a rule is not really a function at all. To give an example, suppose we wrote, “Define a function $f: \mathbf{R} \rightarrow \mathbf{R}$ by specifying that for all real numbers x ,

$f(x)$ is the real number y such that $x^2 + y^2 = 1$.

There are two distinct reasons why this description does not define a function. For almost all values of x , either (1) there is no y that satisfies the given equation or (2) there are two different values of y that satisfy the equation.



Checking Whether a Function Is Well Defined

For instance, when $x = 2$, there is no real number y such that $2^2 + y^2 = 1$, and when $x = 0$, both $y = -1$ and $y = 1$ satisfy the equation $0^2 + y^2 = 1$.

In general, we say that a “function” is **not well defined** if it fails to satisfy at least one of the requirements for being a function.



Example 12 – *A Function That Is Not Well Defined*

We know that \mathbf{Q} represents the set of all rational numbers. Suppose you read that a function $f : \mathbf{Q} \rightarrow \mathbf{Z}$ is to be defined by the formula

$$f\left(\frac{m}{n}\right) = m \text{ for all integers } m \text{ and } n \text{ with } n \neq 0.$$

That is, the integer associated by f to the number $\frac{m}{n}$ is m . Is f well defined? Why?



Example 12 – *Solution*

The function f is not well defined.

The reason is that fractions have more than one representation as quotients of integers.

For instance, $\frac{1}{2} = \frac{3}{6}$. Now if f were a function, then the definition of a function would imply that $f\left(\frac{1}{2}\right) = \left(\frac{3}{6}\right)$ since $\frac{1}{2} = \frac{3}{6}$.



Example 12 – *Solution*

cont'd

But applying the formula for f , you find that

$$f\left(\frac{1}{2}\right) = 1 \quad \text{and} \quad f\left(\frac{3}{6}\right) = 3,$$

and so

$$f\left(\frac{1}{2}\right) \neq f\left(\frac{3}{6}\right).$$

This contradiction shows that f is not well defined and, therefore, is not a function.



Checking Whether a Function Is Well Defined

Note that the phrase *well-defined function* is actually redundant; for a function to be well defined really means that it is worthy of being called a function.



Functions Acting on Sets

Functions Acting on Sets

Given a function from a set X to a set Y , you can consider the set of images in Y of all the elements in a subset of X and the set of inverse images in X of all the elements in a subset of Y .

• Definition

If $f: X \rightarrow Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

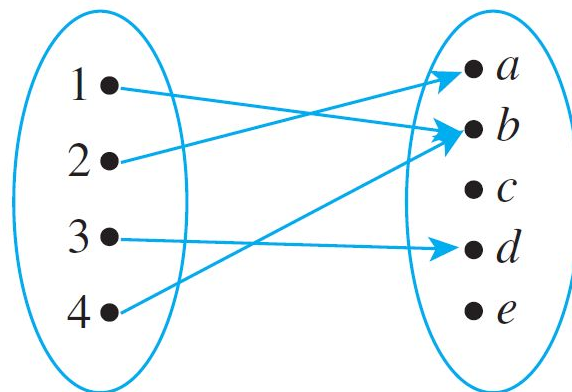
and

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of A** , and $f^{-1}(C)$ is called the **inverse image of C** .

Example 13 – *The Action of a Function on Subsets of a Set*

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$, and define $F : X \rightarrow Y$ by the following arrow diagram:



Let $A = \{1, 4\}$, $C = \{a, b\}$, and $D = \{c, e\}$. Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.



Example 13 – *Solution*

$$F(A) = \{b\}$$

$$F(X) = \{a, b, d\}$$

$$F^{-1}(C) = \{1, 2, 4\}$$

$$F^{-1}(D) = \emptyset$$